ALAN BROMBORSKY

AN
INTRODUCTION
TO
GEOMETRIC
ALGEBRA
AND
CALCULUS

An Introduction to Geometric Algebra and Calculus

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November 15, 2014

Introduction

Geometric algebra is the Clifford algebra of a finite dimensional vector space over real scalars cast in a form most appropriate for physics and engineering. This was done by David Hestenes (Arizona State University) in the 1960's. From this start he developed the geometric calculus whose fundamental theorem includes the generalized Stokes theorem, the residue theorem, and new integral theorems not realized before. Hestenes likes to say he was motivated by the fact that physicists and engineers did not know how to multiply vectors.

Researchers at Arizona State and Cambridge have applied these developments to classical mechanics,quantum mechanics, general relativity (gauge theory of gravity), projective geometry, conformal geometry, etc.

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Chapter 1

Basic Geometric Algebra

1.1 Axioms of Geometric Algebra

Let $\mathcal{V}(p,q)$ be a finite dimensional vector space of signature (p,q)[1] over $\Re$. Then $\forall a,b,c \in \mathcal{V}$ there exists a geometric product with the properties -

$$
\begin{aligned}
(ab)c &= a(bc) \\
a(b+c) &= ab+ac \\
(a+b)c &= ac+bc \\
aa &\in \Re
\end{aligned}
$$

If $a^2 \neq 0$ then $a^{-1} = \dfrac{1}{a^2}a$.

1.2 Why Learn This Stuff?

The geometric product of two (or more) vectors produces something "new" like the $\sqrt{-1}$ with respect to real numbers or vectors with respect to scalars. It must be studied in terms of its effect on vectors and in terms of its symmetries. It is worth the effort. Anything that makes understanding rotations in a N dimensional space simple is worth the effort! Also, if one proceeds on to geometric calculus many diverse areas in mathematics are unified and many areas of physics and engineering are greatly simplified.

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[1] To be completely general we would have to consider $\mathcal{V}(p,q,r)$ where the dimension of the vector space is $n = p+q+r$ and p, q, and r are the number of basis vectors respectively with positive, negative and zero squares.

1.3 Inner, $\cdot$, and outer, $\wedge$, product of two vectors and their basic properties

The inner (dot) and outer (wedge) products of two vectors are defined by

$$a \cdot b \equiv \frac{1}{2}\left(ab + ba\right) \tag{1.1}$$

$$a \wedge b \equiv \frac{1}{2}\left(ab - ba\right) \tag{1.2}$$

$$ab = a \cdot b + a \wedge b \tag{1.3}$$

$$a \wedge b = -b \wedge a \tag{1.4}$$

$$\begin{gathered} c = a + b \\ c^2 = \left(a + b\right)^2 \\ c^2 = a^2 + ab + ba + b^2 \\ 2a \cdot b = c^2 - a^2 - b^2 \\ a \cdot b \in \Re \end{gathered} \tag{1.5}$$

$$a \cdot b = |a|\,|b| \cos\left(\theta\right) \text{ if } a^2, b^2 > 0 \tag{1.6}$$

Orthogonal vectors are defined by $a \cdot b = 0$. For orthogonal vectors $a \wedge b = ab$. Now compute $\left(a \wedge b\right)^2$

$$\begin{aligned} \left(a \wedge b\right)^2 &= -\left(a \wedge b\right)\left(b \wedge a\right) && (1.7) \\ &= -\left(ab - a \cdot b\right)\left(ba - a \cdot b\right) && (1.8) \\ &= -\left(abba - \left(a \cdot b\right)\left(ab + ba\right) + \left(a \cdot b\right)^2\right) && (1.9) \\ &= -\left(a^2b^2 - \left(a \cdot b\right)^2\right) && (1.10) \\ &= -a^2b^2\left(1 - \cos^2\left(\theta\right)\right) && (1.11) \\ &= -a^2b^2 \sin^2\left(\theta\right) && (1.12) \end{aligned}$$

Thus in a Euclidean space, $a^2, b^2 > 0$, $\left(a \wedge b\right)^2 \leq 0$ and $a \wedge b$ is proportional to $\sin\left(\theta\right)$. If $e_{\|}$ and $e_{\perp}$ are any two orthonormal unit vectors in a Euclidean space then $\left(e_{\|}e_{\perp}\right)^2 = -1$. Who needs the $\sqrt{-1}$?

1.4 Outer, ∧, product for r Vectors in terms of the geometric product

Define the outer product of r vectors to be ($\varepsilon^{i_1\dots i_r}_{1\dots r}$ is the mixed permutation symbol)

$$a_1 \wedge \dots \wedge a_r \equiv \frac{1}{r!} \sum_{i_1,\dots,i_r} \varepsilon^{i_1\dots i_r}_{1\dots r} a_{i_1} \dots a_{i_r} \tag{1.13}$$

Thus

$$\begin{aligned} a_1 \wedge \dots \wedge (a_j + b_j) \wedge \dots \wedge a_r = \\ a_1 \wedge \dots \wedge a_j \wedge \dots \wedge a_r + a_1 \wedge \dots \wedge b_j \wedge \dots \wedge a_r \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} a_1 \wedge \dots \wedge a_j \wedge a_{j+1} \wedge \dots \wedge a_r = \\ - a_1 \wedge \dots \wedge a_{j+1} \wedge a_j \wedge \dots \wedge a_r \end{aligned} \tag{1.15}$$

The outer product of r vectors is called a blade of grade r.

1.5 Alternate Definition of Outer, ∧, product for r Vectors

Let $e_1, e_2, \dots, e_r$ be an orthogonal basis for the set of linearly independent vectors $a_1, a_2, \dots, a_r$ so that we can write

$$a_i = \sum_j \alpha_{ij} e_j \tag{1.16}$$

Then

$$\begin{aligned} a_1 a_2 \dots a_r &= \left(\sum_{j_1} \alpha_{1j_1} e_{j_1}\right) \left(\sum_{j_2} \alpha_{2j_2} e_{j_2}\right) \dots \left(\sum_{j_r} \alpha_{rj_r} e_{j_r}\right) \\ &= \sum_{j_1,\dots,j_r} \alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{rj_r} e_{j_1} e_{j_2} \dots e_{j_r} \end{aligned} \tag{1.17}$$

Now define a blade of grade n as the geometric product of n orthogonal vectors. Thus the product $e_{j_1} e_{j_2} \dots e_{j_r}$ in equation 1.17 could be a blade of grade r, $r-2$, $r-4$, etc. depending upon the number of repeated factors.

If there are no repeated factors in the product we have that

$$e_{j_1}\dots e_{j_r} = \varepsilon^{j_1\dots j_r}_{1\dots r} e_1\dots e_r \tag{1.18}$$

Due to the fact that interchanging two adjacent orthogonal vectors in the geometric product will reverse the sign of the product and we can define the outer product of r vectors as

$$\begin{aligned} a_1\wedge\dots\wedge a_r &= \sum_{j_1,\dots,j_r} \varepsilon^{j_1\dots j_r}_{1\dots r}\alpha_{1j_1}\dots\alpha_{rj_r}e_1\dots e_r &(1.19)\\ &= \det(\alpha)\, e_1\dots e_r &(1.20)\end{aligned}$$

Thus the outer product of r independent vectors is the part of the geometric product of the r vectors that is of grade r. Equation 1.19 is equivalent to equation 1.13. This can be proved by substituting equation 1.17 into equation 1.13 to get

$$\begin{aligned} a_1\wedge\dots\wedge a_r &= \frac{1}{r!}\sum_{i_1,\dots,i_r}\sum_{j_1,\dots,j_r}\varepsilon^{i_1\dots i_r}_{1\dots r}\alpha_{i_1j_1}\dots\alpha_{i_rj_r}e_{j_1}\dots e_{j_r} &(1.21)\\ &= \frac{1}{r!}\sum_{i_1,\dots,i_r}\sum_{j_1,\dots,j_r}\varepsilon^{i_1\dots i_r}_{1\dots r}\varepsilon^{j_1\dots j_r}_{1\dots r}\alpha_{i_1j_1}\dots\alpha_{i_rj_r}e_1\dots e_r &(1.22)\\ &= \frac{1}{r!}\sum_{j_1,\dots,j_r}\varepsilon^{j_1\dots j_r}_{1\dots r}\varepsilon^{j_1\dots j_r}_{1\dots r}\det(\alpha)\, e_1\dots e_r &(1.23)\\ &= \det(\alpha)\, e_1\dots e_r &(1.24)\end{aligned}$$

We go from equation 1.22 to equation 1.23 by noting that $\sum_{i_1,\dots,i_r}\varepsilon^{i_1\dots i_r}_{1\dots r}\alpha_{i_1j_1}\dots\alpha_{i_rj_r}$ is just $\det(\alpha)$ with the columns permuted. Multiplying $\det(\alpha)$ by $\varepsilon^{j_1\dots j_r}_{1\dots r}$ gives the correct sign for the determinant with the columns permuted.

If $e_1,\dots,e_n$ is an orthonormal basis for vector space the unit psuedoscalar is defined as

$$I = e_1\dots e_n \tag{1.25}$$

In equation 1.24 let $r = n$ and the $a_1,\dots,a_n$ be another orthonormal basis for the vector space. Then we may write

$$a_1\dots a_n = \det(\alpha)\, e_1\dots e_n \tag{1.26}$$

Since both the a's and the e's form orthonormal bases the matrix α is orthogonal and $\det(\alpha) = \pm 1$. All psuedoscalars for the vector space are identical to within a scale factor of ±1.[2] Likewise $a_1\wedge\dots\wedge a_n$ is equal to I times a scale factor.

---

[2] It depends only upon the ordering of the basis vectors.

1.6 Useful Relation's

1. For a set of r orthogonal vectors, $e_1, \ldots, e_r$

$$e_1 \wedge \ldots \wedge e_r = e_1 \ldots e_r \tag{1.27}$$

2. For a set of r linearly independent vectors, $a_1, \ldots, a_r$, there exists a set of r orthogonal vectors, $e_1, \ldots, e_r$, such that

$$a_1 \wedge \ldots \wedge a_r = e_1 \ldots e_r \tag{1.28}$$

   If the vectors, $a_1, \ldots, a_r$, are not linearly independent then

$$a_1 \wedge \ldots \wedge a_r = 0 \tag{1.29}$$

The product $a_1 \wedge \ldots \wedge a_r$ is call a "blade" of grade r. The dimension of the vector space is the highest grade any blade can have.

1.7 Projection Operator

A multivector, the basic element of the geometric algebra, is made of of a sum of scalars, vectors, blades. A multivector is homogeneous (pure) if all the blades in it are of the same grade. The grade of a scalar is 0 and the grade of a vector is 1. The general multivector A is decomposed with the grade projection operator $\left\langle A \right\rangle_r$ as (N is dimension of the vector space):

$$A = \sum_{r=0}^{N} \left\langle A \right\rangle_r \tag{1.30}$$

As an example consider ab, the product of two vectors. Then

$$ab = \left\langle ab \right\rangle_0 + \left\langle ab \right\rangle_2 \tag{1.31}$$

We define $\left\langle A \right\rangle \equiv \left\langle A \right\rangle_0$ for any multivector A

1.8 Basis Blades

The geometric algebra of a vector space, $\mathcal{V}(p,q)$, is denoted $\mathcal{G}(p,q)$ or $\mathcal{G}(\mathcal{V})$ where (p,q) is the signature of the vector space (first p unit vectors square to $+1$ and next q unit vectors square to -1, dimension of the space is $p+q$). Examples are:

p	q	Type of Space
3	0	3D Euclidean
1	3	Relativistic Space Time
4	1	3D Conformal Geometry

If the orthonormal basis set of the vector space is $e_1, \ldots, e_N$, the basis of the geometric algebra (multivector space) is formed from the geometric products (since we have chosen an orthonormal basis, $ei^2 = \pm 1$) of the basis vectors. For grade r multivectors the basis blades are all the combinations of basis vectors products taken r at a time from the set of N vectors. Thus the number basis blades of rank r are $\binom{N}{r}$, the binomial expansion coefficient and the total dimension of the multivector space is the sum of $\binom{N}{r}$ over r which is 2^N.

1.8.1 $\mathcal{G}(3,0)$ Geometric Algebra (Euclidian Space)

The basis blades for $\mathcal{G}(3,0)$ are:

Grade			
0	1	2	3
1	e_1	e_1e_2	$e_1e_2e_3$
	e_2	e_1e_3	
	e_3	e_2e_3	

The multiplication table for the $\mathcal{G}(3,0)$ basis blades is

	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
1	1	e_1	e_2	e_3	e_1e_2	e_1e_3	e_2e_3	$e_1e_2e_3$
e_1	e_1	1	e_1e_2	e_1e_3	e_2	e_3	$e_1e_2e_3$	e_2e_3
e_2	e_2	$-e_1e_2$	1	e_2e_3	$-e_1$	$-e_1e_2e_3$	e_3	$-e_1e_3$
e_3	e_3	$-e_1e_3$	$-e_2e_3$	1	$e_1e_2e_3$	$-e_1$	$-e_2$	e_1e_2
e_1e_2	e_1e_2	$-e_2$	e_1	$e_1e_2e_3$	-1	$-e_2e_3$	e_1e_3	$-e_3$
e_1e_3	e_1e_3	$-e_3$	$-e_1e_2e_3$	e_1	e_2e_3	-1	$-e_1e_2$	e_2
e_2e_3	e_2e_3	$e_1e_2e_3$	$-e_3$	e_2	$-e_1e_3$	e_1e_2	-1	$-e_1$
$e_1e_2e_3$	$e_1e_2e_3$	e_2e_3	$-e_1e_3$	e_1e_2	$-e_3$	e_2	$-e_1$	-1

Note that the squares of all the grade 2 and 3 basis blades are -1. The highest rank basis blade (in this case $e_1e_2e_3$) is usually denoted by I and is called the pseudoscalar.

1.8.2 $\mathcal{G}(1,3)$ Geometric Algebra (Spacetime)

The multiplication table for the $\mathcal{G}(1,3)$ basis blades is

	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
1	1	γ_0	γ_1	γ_2	γ_3	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_1\gamma_2$
γ_0	γ_0	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	γ_1	γ_2	$\gamma_0\gamma_1\gamma_2$
γ_1	γ_1	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	γ_0	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_2$
γ_2	γ_2	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_0	γ_1
γ_3	γ_3	$-\gamma_0\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_2\gamma_3$	-1	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
$\gamma_0\gamma_1$	$\gamma_0\gamma_1$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	1	$-\gamma_1\gamma_2$	$-\gamma_0\gamma_2$
$\gamma_0\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	1	$\gamma_0\gamma_1$
$\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	-1
$\gamma_0\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
$\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$
$\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	γ_3	$-\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$
$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	$-\gamma_1$	$-\gamma_0$
$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$\gamma_1\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$
$\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$
$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$

	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
1	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$
γ_0	γ_3	$\gamma_0\gamma_1\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$
γ_1	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$
γ_2	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_1$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_3$	$\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_3$
γ_3	γ_0	γ_1	γ_2	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_1$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2$
$\gamma_0\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$\gamma_0\gamma_1\gamma_2\gamma_3$	γ_2	γ_3	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_2\gamma_3$	$\gamma_2\gamma_3$
$\gamma_0\gamma_2$	$-\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_2\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_1$	$\gamma_1\gamma_2\gamma_3$	γ_3	$\gamma_0\gamma_1\gamma_3$	$-\gamma_1\gamma_3$
$\gamma_1\gamma_2$	$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	$-\gamma_3$	$-\gamma_0\gamma_3$
$\gamma_0\gamma_3$	1	$\gamma_0\gamma_1$	$\gamma_0\gamma_2$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2$
$\gamma_1\gamma_3$	$-\gamma_0\gamma_1$	-1	$\gamma_1\gamma_2$	$-\gamma_0\gamma_2\gamma_3$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	γ_2	$\gamma_0\gamma_2$
$\gamma_2\gamma_3$	$-\gamma_0\gamma_2$	$-\gamma_1\gamma_2$	-1	$\gamma_0\gamma_1\gamma_3$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$-\gamma_1$	$-\gamma_0\gamma_1$
$\gamma_0\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_2\gamma_3$	$-\gamma_0\gamma_1\gamma_3$	-1	$\gamma_2\gamma_3$	$-\gamma_1\gamma_3$	$-\gamma_0\gamma_3$	$-\gamma_3$
$\gamma_0\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_0$	$\gamma_0\gamma_1\gamma_2$	$-\gamma_2\gamma_3$	-1	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	γ_2
$\gamma_0\gamma_2\gamma_3$	$-\gamma_2$	$-\gamma_0\gamma_1\gamma_2$	$-\gamma_0$	$\gamma_1\gamma_3$	$-\gamma_1\gamma_2$	-1	$-\gamma_0\gamma_1$	$-\gamma_1$
$\gamma_1\gamma_2\gamma_3$	$\gamma_0\gamma_1\gamma_2$	γ_2	$-\gamma_1$	$\gamma_0\gamma_3$	$-\gamma_0\gamma_2$	$\gamma_0\gamma_1$	1	$-\gamma_0$
$\gamma_0\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2$	$\gamma_0\gamma_2$	$-\gamma_0\gamma_1$	γ_3	$-\gamma_2$	γ_1	γ_0	-1

1.9 Reflections

We wish to show that $a, v \in \mathcal{V} \rightarrow ava \in \mathcal{V}$ and v is reflected about a if $a^2 = 1$.

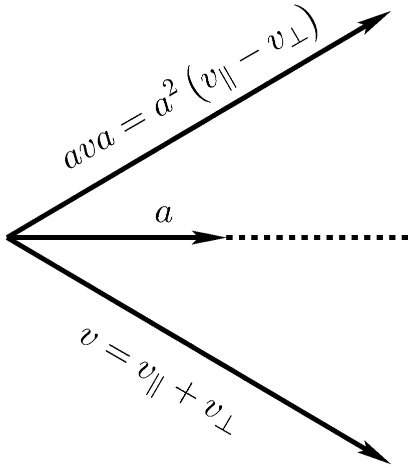


Figure 1.1: Reflection of Vector

1. Decompose $v = v_{\|} + v_{\perp}$ where $v_{\|}$ is the part of v parallel to a and $v_{\perp}$ is the part perpendicular to a.
2. $av = av_{\|} + av_{\perp} = v_{\|}a - v_{\perp}a$ since a and $v_{\perp}$ are orthogonal.
3. $ava = a^2 \left(v_{\|} - v_{\perp}\right)$ is a vector since a^2 is a scalar.
4. ava is the reflection of v about the direction of a if $a^2 = 1$.
5. Thus $a_1 \ldots a_r v a_r \ldots a_1 \in \mathcal{V}$ and produces a composition of reflections of v if $a_1^2 = \cdots = a_r^2 = 1$.

1.10 Rotations

1.10.1 Definitions

First define the reverse of a product of vectors. If $R = a_1 \dots a_s$ then the reverse is $R^\dagger = (a_1 \dots a_s)^\dagger = a_r \dots a_1$, the order of multiplication is reversed. Then let $R = ab$ so that

$$RR^\dagger = (ab)(ba) = ab^2a = a^2b^2 = R^\dagger R \tag{1.32}$$

Let $RR^\dagger = 1$ and calculate $\left(RvR^\dagger\right)^2$, where v is an arbitrary vector.

$$\left(RvR^\dagger\right)^2 = RvR^\dagger RvR^\dagger = Rv^2R^\dagger = v^2RR^\dagger = v^2 \tag{1.33}$$

Thus $RvR^\dagger$ leaves the length of v unchanged. Now we must also prove $Rv_1R^\dagger \cdot Rv_2R^\dagger = v_1 \cdot v_2$. Since $Rv_1R^\dagger$ and $Rv_2R^\dagger$ are both vectors we can use the definition of the dot product for two vectors

$$\begin{aligned}
Rv_1R^\dagger \cdot Rv_2R^\dagger &= \frac{1}{2}\left(Rv_1R^\dagger Rv_2R^\dagger + Rv_2R^\dagger Rv_1R^\dagger\right) \\
&= \frac{1}{2}\left(Rv_1v_2R^\dagger + Rv_2v_1R^\dagger\right) \\
&= \frac{1}{2}R\left(v_1v_2 + v_2v_1\right)R^\dagger \\
&= R\left(v_1 \cdot v_2\right)R^\dagger \\
&= v_1 \cdot v_2RR^\dagger \\
&= v_1 \cdot v_2
\end{aligned}$$

Thus the transformation $RvR^\dagger$ preserves both length and angle and must be a rotation. The normal designation for R is a rotor. If we have a series of successive rotations $R_1, R_2, \dots, R_k$ to be applied to a vector v then the result of the k rotations will be

$$R_kR_{k-1} \dots R_1vR_1^\dagger R_2^\dagger \dots R_k^\dagger$$

Since each individual rotation can be written as the geometric product of two vectors, the composition of k rotations can be written as the geometric product of $2k$ vectors. The multivector that results from the geometric product of r vectors is called a **versor** of order r. A composition of rotations is always a versor of even order.

1.10.2 General Rotation

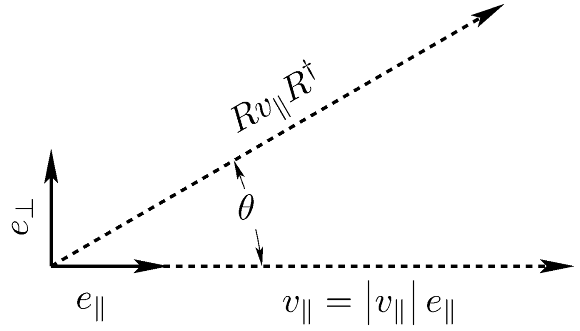
The general rotation can be represented by $R = e^{\frac{\theta}{2}u}$ where u is a unit bivector in the plane of the rotation and θ is the rotation angle in the plane.[3] The two possible non-degenerate cases are $u^2 = \pm 1$

$$e^{\frac{\theta}{2}u} = \left\{ \begin{array}{ccc} \text{(Euclidean plane)} & u^2 = -1: & \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right) \\ \text{(Minkowski plane)} & u^2 = 1: & \cosh\left(\frac{\theta}{2}\right) + u\sinh\left(\frac{\theta}{2}\right) \end{array} \right\} \tag{1.34}$$

Decompose $v = v_{\parallel} + \left(v - v_{\parallel}\right)$ where $v_{\parallel}$ is the projection of v into the plane defined by u. Note that $v - v_{\parallel}$ is orthogonal to all vectors in the u plane. Now let $u = e_{\perp}e_{\parallel}$ where $e_{\parallel}$ is parallel to $v_{\parallel}$ and of course $e_{\perp}$ is in the plane u and orthogonal to $e_{\parallel}$. $v - v_{\parallel}$ anticommutes with $e_{\parallel}$ and $e_{\perp}$ and $v_{\parallel}$ anticommutes with $e_{\perp}$ (it is left to the reader to show $RR^{\dagger} = 1$).

1.10.3 Euclidean Case

For the case of $u^2 = -1$



$Rv_{\parallel}R^{\dagger}$
$e_{\perp}$
θ
$e_{\parallel}$
$v_{\parallel} = \left|v_{\parallel}\right| e_{\parallel}$

Figure 1.2: Rotation of Vector

$$RvR^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sin\left(\frac{\theta}{2}\right)\right)\left(v_{\parallel} + \left(v - v_{\parallel}\right)\right)\left(\cos\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right)$$

Since $v - v_{\parallel}$ anticommutes with $e_{\parallel}$ and $e_{\perp}$ it commutes with R and

$$RvR^{\dagger} = Rv_{\parallel}R^{\dagger} + \left(v - v_{\parallel}\right) \tag{1.35}$$

---

[3] e^A is defined as the Taylor series expansion $e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$ where A is any multivector.

So that we only have to evaluate

$$Rv_{\parallel}R^{\dagger} = \left(\cos\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sin\left(\frac{\theta}{2}\right)\right) v_{\parallel} \left(\cos\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sin\left(\frac{\theta}{2}\right)\right) \tag{1.36}$$

Since $v_{\parallel} = \left|v_{\parallel}\right| e_{\parallel}$

$$Rv_{\parallel}R^{\dagger} = \left|v_{\parallel}\right| \left(\cos\left(\theta\right) e_{\parallel} + \sin\left(\theta\right) e_{\perp}\right) \tag{1.37}$$

and the component of v in the u plane is rotated correctly.

1.10.4 Minkowski Case

For the case of $u^2 = 1$ there are two possibilities, $v_{\parallel}^2 > 0$ or $v_{\parallel}^2 < 0$. In the first case $e_{\parallel}^2 = 1$ and $e_{\perp}^2 = -1$. In the second case $e_{\parallel}^2 = -1$ and $e_{\perp}^2 = 1$. Again $v - v_{\parallel}$ is not affected by the rotation so that we need only evaluate

$$Rv_{\parallel}R^{\dagger} = \left(\cosh\left(\frac{\theta}{2}\right) + e_{\perp}e_{\parallel}\sinh\left(\frac{\theta}{2}\right)\right) v_{\parallel} \left(\cosh\left(\frac{\theta}{2}\right) + e_{\parallel}e_{\perp}\sinh\left(\frac{\theta}{2}\right)\right)$$

Note that in this case $\left|v_{\parallel}\right| = \sqrt{\left|v_{\parallel}^2\right|}$ and

$$Rv_{\parallel}R^{\dagger} = \left\{ \begin{array}{l} v_{\parallel}^2 > 0 : \left|v_{\parallel}\right| \left(\cosh\left(\theta\right) e_{\parallel} + \sinh\left(\theta\right) e_{\perp}\right) \\ v_{\parallel}^2 < 0 : \left|v_{\parallel}\right| \left(\cosh\left(\theta\right) e_{\parallel} - \sinh\left(\theta\right) e_{\perp}\right) \end{array} \right\} \tag{1.38}$$

1.11 Expansion of geometric product and generalization of · and ∧

If A_r and B_s are respectively grade r and s pure grade multivectors then

$$A_rB_s = \left\langle A_rB_s\right\rangle_{|r-s|} + \left\langle A_rB_s\right\rangle_{|r-s|+2} + \cdots + \left\langle A_rB_s\right\rangle_{\min(r+s,2N-(r+s))} \tag{1.39}$$

$$A_r \cdot B_s \equiv \left\langle A_rB_s\right\rangle_{|r-s|} \tag{1.40}$$

$$A_r \wedge B_s \equiv \left\langle A_rB_s\right\rangle_{r+s} \tag{1.41}$$

Thus if $r + s > N$ then $A_r \wedge B_s = 0$, also note that these formulas are the most efficient way of calculating $A_r \cdot B_s$ and $A_r \wedge B_s$. Using equations 1.28 and 1.39 we can prove that for a vector a and a grade r multivector B_r

$$a \cdot B_r = \frac{1}{2}\left(aB_r - \left(-1\right)^r B_ra\right) \tag{1.42}$$

$$a \wedge B_r = \frac{1}{2}\left(aB_r + (-1)^r B_r a\right) \tag{1.43}$$

If equations 1.42 and 1.43 are true for a grade r blade they are also true for a grade r multivector (superposition of grade r blades). By equation 1.28 let $B_r = e_1 \dots e_r$ where the $e's$ are orthogonal and expand a

$$a = a_\perp + \sum_{j=1}^{r} \alpha_j e_j \tag{1.44}$$

where $a_\perp$ is orthogonal to all the $e's$. Then[4]

$$\begin{aligned} aB_r &= \sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r + a_\perp e_1 \dots e_r \\ &= a \cdot B_r + a \wedge B_r \end{aligned} \tag{1.45}$$

Now calculate

$$\begin{aligned} B_r a &= \sum_{j=1}^{r} (-1)^{r-j} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - (-1)^{r-1} a_\perp e_1 \dots e_r \\ &= (-1)^{r-1} \left(\sum_{j=1}^{r} (-1)^{j-1} \alpha_j e_j^2 e_1 \cdots \breve{e}_j \cdots e_r - a_\perp e_1 \dots e_r \right) \\ &= (-1)^{r-1} \left(a \cdot B_r - a \wedge B_r \right) \end{aligned} \tag{1.46}$$

Adding and subtracting equations 1.45 and 1.46 gives equations 1.42 and 1.43.

1.12 Duality and the Pseudoscalar

If $e_1, \dots, e_n$ is an orthonormal basis for the vector space the the pseudoscalar I is defined by

$$I = e_1 \dots e_n \tag{1.47}$$

Since one can transform one orthonormal basis to another by an orthogonal transformation the I's for all orthonormal bases are equal to within a ±1 scale factor with depends on the ordering of the basis vectors. If A_r is a pure r grade multivector ($A_r = \langle A_r \rangle_r$) then

$$A_r I = \langle A_r I \rangle_{n-r} \tag{1.48}$$

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[4] $e_1 \dots e_{j-1} \breve{e}_j e_{j+1} \dots e_r = e_1 \dots e_{j-1} e_{j+1} \dots e_r$

or A_rI is a pure $n-r$ grade multivector. Further by the symmetry properties of I we have

$$IA_r = (-1)^{(n-1)r} A_rI \tag{1.49}$$

I can also be used to exchange the $\cdot$ and $\wedge$ products as follows using equations 1.42 and 1.43

$$\begin{aligned} a\cdot(A_rI) &= \frac{1}{2}\left(aA_rI - (-1)^{n-r} A_rIa\right) && (1.50)\\ &= \frac{1}{2}\left(aA_rI - (-1)^{n-r}(-1)^{n-1} A_raI\right) && (1.51)\\ &= \frac{1}{2}\left(aA_r + (-1)^{r} A_ra\right) I && (1.52)\\ &= (a\wedge A_r)\, I && (1.53)\end{aligned}$$

More generally if A_r and B_s are pure grade multivectors with $r+s \le n$ we have using equation 1.40 and 1.48

$$\begin{aligned} A_r\cdot(B_sI) &= \left\langle A_rB_sI\right\rangle_{|r-(n-s)|} && (1.54)\\ &= \left\langle A_rB_sI\right\rangle_{n-(r+s)} && (1.55)\\ &= \left\langle A_rB_s\right\rangle_{r+s} I && (1.56)\\ &= (A_r\wedge B_s)\, I && (1.57)\end{aligned}$$

Finally we can relate I to $I^\dagger$ by

$$I^\dagger = (-1)^{\frac{n(n-1)}{2}} I \tag{1.58}$$

1.13 Reciprocal Frames

Let $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ be a set of linearly independent vectors that span the vector space that are not necessarily orthogonal. These vectors define the frame (frame vectors are shown in bold face since they are almost always associated with a particular coordinate system) with volume element

$$E_n \equiv \boldsymbol{e}_1\wedge\dots\wedge\boldsymbol{e}_n \tag{1.59}$$

So that $E_n \propto I$. The reciprocal frame is the set of vectors $\boldsymbol{e}^1,\dots,\boldsymbol{e}^n$ that satisfy the relation

$$\boldsymbol{e}^i\cdot\boldsymbol{e}_j = \delta^i_j,\quad \forall i,j = 1,\dots,n \tag{1.60}$$

The $\boldsymbol{e}^i$ are constructed as follows

$$\boldsymbol{e}^j = (-1)^{j-1}\,\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \breve{\boldsymbol{e}}_j \wedge \ldots \wedge \boldsymbol{e}_n E_n^{-1} \tag{1.61}$$

So that the dot product is (using equation 1.53 since $E_n^{-1} \propto I$)

$$\begin{aligned}
\boldsymbol{e}_i \cdot \boldsymbol{e}^j &= (-1)^{j-1}\,\boldsymbol{e}_i \cdot \left(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \breve{\boldsymbol{e}}_j \wedge \ldots \wedge \boldsymbol{e}_n E_n^{-1}\right) && (1.62)\\
&= (-1)^{j-1}\left(\boldsymbol{e}_i \wedge \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \breve{\boldsymbol{e}}_j \wedge \ldots \wedge \boldsymbol{e}_n\right) E_n^{-1} && (1.63)\\
&= 0, \quad \forall i \neq j && (1.64)
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{e}_1 \cdot \boldsymbol{e}^1 &= \boldsymbol{e}_1 \cdot \left(\boldsymbol{e}_2 \wedge \ldots \wedge \boldsymbol{e}_n E_n^{-1}\right) && (1.65)\\
&= \left(\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \ldots \wedge \boldsymbol{e}_n\right) E_n^{-1} && (1.66)\\
&= 1 && (1.67)
\end{aligned}$$

1.14 Coordinates

The reciprocal frame can be used to develop a coordinate representation for multivectors in an arbitrary frame $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ with reciprocal frame $\boldsymbol{e}^1, \ldots, \boldsymbol{e}^n$. Since both the frame and it's reciprocal span the base vector space we can write any vector a in the vector space as

$$a = a^i \boldsymbol{e}_i = a_i \boldsymbol{e}^i \tag{1.68}$$

where if an index such as i is repeated it is assumes that the terms with the repeated index will be summed from 1 to n. Using that $\boldsymbol{e}_i \cdot \boldsymbol{e}^j = \delta_i^j$ we have

$$\begin{aligned}
a_i &= a \cdot \boldsymbol{e}_i && (1.69)\\
a^i &= a \cdot \boldsymbol{e}^i && (1.70)
\end{aligned}$$

In tensor notation a_i would be the covariant representation and a^i the contravariant representation of the vector a. Now consider the case of grade 2 and grade 3 blades:

$$\begin{aligned}
\boldsymbol{e}^i \cdot (a \wedge b) &= a \cdot \boldsymbol{e}^i b - b \cdot \boldsymbol{e}^i a\\
\boldsymbol{e}_i \left(a \cdot \boldsymbol{e}^i b - b \cdot \boldsymbol{e}^i a\right) &= ab - ba = 2a \wedge b\\
\boldsymbol{e}^i \cdot (a \wedge b \wedge c) &= a \cdot \boldsymbol{e}^i b \wedge c - b \cdot \boldsymbol{e}^i a \wedge c + c \cdot \boldsymbol{e}^i a \wedge b\\
\boldsymbol{e}_i \left(a \cdot \boldsymbol{e}^i b \wedge c - b \cdot \boldsymbol{e}^i a \wedge c + c \cdot \boldsymbol{e}^i a \wedge b\right) &= ab \wedge c - ba \wedge c + ca \wedge b = 3a \wedge b \wedge c
\end{aligned}$$

for an r-blade A_r we have (the proof is left to the reader)

$$\boldsymbol{e}_i\boldsymbol{e}^i \cdot A_r = rA_r \tag{1.71}$$

Since $\boldsymbol{e}_i\boldsymbol{e}^i = n$ we have

$$\boldsymbol{e}_i\boldsymbol{e}^i \wedge A_r = \boldsymbol{e}_i\left(\boldsymbol{e}^i A_r - \boldsymbol{e}^i \cdot A_r\right) = (n-r)\,A_r \tag{1.72}$$

Flipping $\boldsymbol{e}^i$ and A_r in equations 1.71 and 1.72 and subtracting equation 1.71 from 1.72 gives

$$\boldsymbol{e}_i A_r \boldsymbol{e}^i = (-1)^r\,(n-2r)\,A_r \tag{1.73}$$

In Hestenes and Sobczyk (3.14) it is proved that

$$\left(\boldsymbol{e}^{k_r} \wedge \ldots \wedge \boldsymbol{e}^{k_1}\right) \cdot \left(\boldsymbol{e}_{j_1} \wedge \ldots \wedge \boldsymbol{e}_{j_r}\right) = \delta^{j_1}_{k_1}\delta^{j_2}_{k_2}\ldots\delta^{j_r}_{k_r} \tag{1.74}$$

so that the general multivector A can be expanded in terms of the blades of the frame and reciprocal frame as

$$A = \sum_{i<j<\cdots<k} A_{ij\cdots k}\boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \cdots \wedge \boldsymbol{e}^k \tag{1.75}$$

where

$$A_{ij\cdots k} = \left(\boldsymbol{e}_k \wedge \cdots \wedge \boldsymbol{e}_j \wedge \boldsymbol{e}_i\right) \cdot A \tag{1.76}$$

The components $A_{ij\cdots k}$ are totally antisymmetric on all indices and are usually referred to as the components of an *antisymmetric tensor*.

1.15 Linear Transformations

1.15.1 Definitions

Let f be a linear transformation on a vector space $f : \mathcal{V} \rightarrow \mathcal{V}$ with $f(\alpha a + \beta b) = \alpha f(a) + \beta f(b)$ $\forall a, b \in \mathcal{V}$ and $\alpha, \beta \in \Re$. Then define the action of f on a blade of the geometric algebra by

$$f(a_1 \wedge \ldots \wedge a_r) = f(a_1) \wedge \ldots \wedge f(a_1) \tag{1.77}$$

and the action of f on any two $A, B \in \mathcal{G}(\mathcal{V})$ by

$$f(\alpha A + \beta B) = \alpha f(A) + \beta f(B) \tag{1.78}$$

Since any multivector A can be expanded as a sum of blades $f(A)$ is defined. This has many consequences. Consider the following definition for the determinant of f, $\det(f)$.

$$f(I) = \det(f)\, I \tag{1.79}$$

First show that this definition is equivalent to the standard definition of the determinant (again $e_1, \dots, e_N$ is an orthonormal basis for $\mathcal{V}$).

$$f(e_r) = \sum_{s=1}^{N} a_{rs} e_s \tag{1.80}$$

Then

$$\begin{aligned} f(I) &= \left(\sum_{s_1=1}^{N} a_{1s_1} e_s\right) \wedge \dots \wedge \left(\sum_{s_N=1}^{N} a_{Ns_N} e_s\right) \\ &= \sum_{s_1,\dots,s_N} a_{1s_1} \dots a_{Ns_N} e_{s_1} \dots e_{s_N} \end{aligned} \tag{1.81}$$

But

$$e_{s_1} \dots e_{s_N} = \varepsilon^{s_1 \dots s_N}_{1 \dots N} e_1 \dots e_N \tag{1.82}$$

so that

$$f(I) = \sum_{s_1,\dots,s_N} \varepsilon^{s_1 \dots s_N}_{1 \dots N} a_{1s_1} \dots a_{Ns_N} I \tag{1.83}$$

or

$$\det(f) = \sum_{s_1,\dots,s_N} \varepsilon^{s_1 \dots s_N}_{1 \dots N} a_{1s_1} \dots a_{Ns_N} \tag{1.84}$$

which is the standard definition. Now compute the determinant of the product of the linear transformations f and g

$$\begin{aligned} \det(fg)\, I &= fg(I) \\ &= f(g(I)) \\ &= f(\det(g)\, I) \\ &= \det(g)\, f(I) \\ &= \det(g) \det(f)\, I \end{aligned} \tag{1.85}$$

or

$$\det(fg) = \det(f) \det(g) \tag{1.86}$$

Do you have any idea of how miserable that is to prove from the standard definition of determinant?

1.15.2 Adjoint

If F is linear transformation and a and b are two arbitrary vectors the adjoint function, $\overline{F}$, is defined by

$$a\cdot\overline{F}\left(b\right)=b\cdot F\left(a\right)\tag{1.87}$$

From the definition the adjoint is also a linear transformation. For an arbitrary frame $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ we have

$$\boldsymbol{e}_i\cdot\overline{F}\left(a\right)=a\cdot F\left(\boldsymbol{e}_i\right)\tag{1.88}$$

So that we can explicitly construct the adjoint as

$$\begin{aligned}\overline{F}\left(a\right)&=\boldsymbol{e}^i\left(\boldsymbol{e}_i\cdot\overline{F}\left(a\right)\right)\\&=\boldsymbol{e}^i\left(a\cdot F\left(\boldsymbol{e}_i\right)\right)\\&=\boldsymbol{e}^i\left(F\left(\boldsymbol{e}_i\right)\cdot\boldsymbol{e}^j\right)a_j\end{aligned}\tag{1.89}$$

so that $\overline{F}_{ij}=F\left(\boldsymbol{e}_i\right)\cdot\boldsymbol{e}^j$ is the matrix representation of $\overline{F}$ for the $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ frame. However

$$F\left(a\right)=\boldsymbol{e}^i\left(F\left(\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_i\right)a_j\tag{1.90}$$

so that the matrix representation of F is $F_{ij}=F\left(\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_i$. If the $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ are orthonormal then $\boldsymbol{e}_j=\boldsymbol{e}^j$ for all j and $\overline{F}_{ij}=F_{ji}$ exactly the same as the adjoint in matrices.

Other basic properties of the adjoint are:

$$\overline{\overline{F}}\left(a\right)=\boldsymbol{e}^i a\cdot\overline{F}\left(\boldsymbol{e}_i\right)=\boldsymbol{e}^i\boldsymbol{e}_i\cdot F\left(a\right)=F\left(a\right)\tag{1.91}$$

and

$$\begin{aligned}\overline{FG}\left(a\right)&=\boldsymbol{e}^i\left(\boldsymbol{e}_i\cdot\overline{FG}\left(a\right)\right)\\&=\boldsymbol{e}^i\left(a\cdot F\left(G\left(\boldsymbol{e}_i\right)\right)\right)\\&=\boldsymbol{e}^i\left(\overline{F}\left(a\right)\cdot G\left(\boldsymbol{e}_i\right)\right)\\&=\boldsymbol{e}^i\left(\boldsymbol{e}_i\cdot\overline{G}\left(\overline{F}\left(a\right)\right)\right)\\&=\overline{G}\left(\overline{F}\left(a\right)\right)\end{aligned}\tag{1.92}$$

so that $\overline{\overline{F}}=F$ and $\overline{FG}=\overline{G}\,\overline{F}$. A symmetric function is one where $F=\overline{F}$. As an example consider $F\overline{F}$

$$\overline{F\overline{F}}=\overline{\overline{F}}\,\overline{F}=F\overline{F}\tag{1.93}$$

1.15.3 Inverse

Another linear algebraic relation in geometric algebra is

$$f^{-1}(A) = \frac{I\overline{f}\left(I^{-1}A\right)}{\det(f)} \quad \forall A \in \mathcal{G}(\mathcal{V}) \tag{1.94}$$

where $\overline{f}$ is the adjoint transformation defined by $a \cdot \overline{f}(b) = b \cdot f(a)\ \forall a, b \in \mathcal{V}$ and you have an explicit formula for the inverse of a linear transformation!

1.16 Commutator Product

The commutator product of two multivectors A and B is defined as

$$A \times B \equiv \frac{1}{2}(AB - BA) \tag{1.95}$$

An important theorem for the commutator product is that for a grade 2 multivector, $A_2 = \langle A \rangle_2$, and a grade r multivector $B_r = \langle B \rangle_r$ we have

$$A_2 B_r = A_2 \wedge B_r + A_2 \times B_r + A_2 \cdot B_r \tag{1.96}$$

From the geometric product grade expansion for multivectors we have

$$A_2 B_r = \langle A_2 B_r \rangle_{r+2} + \langle A_2 B_r \rangle_r + \langle A_2 B_r \rangle_{|r-2|} \tag{1.97}$$

Thus we must show that

$$\langle A_2 B_r \rangle_r = A_2 \times B_r \tag{1.98}$$

Let $e_1, \dots, e_n$ be an orthogonal set for the vector space where $B_r = e_1 \dots e_r$ and $A_2 = \sum\limits_{l<m=2}^{n} \alpha_{lm} e_l e_m$ so we can write

$$A_2 \times B_r = \left(\sum_{l<m=2}^{n} \alpha_{lm} e_l e_m \right) \times (e_1 \dots e_r) \tag{1.99}$$

Now consider the following three cases

1. l and $m > r$ where $e_l e_m e_1 \dots e_r = e_1 \dots e_r e_l e_m$
2. $l \le r$ and $m > r$ where $e_l e_m e_1 \dots e_r = -e_1 \dots e_r e_l e_m$

3. l and $m \leq r$ where $e_l e_m e_1 \ldots e_r = e_1 \ldots e_r e_l e_m$

For case 1 and 3 $e_l e_m$ commute with B_r and the contribution to the commutator product is zero. In case 3 $e_l e_m$ anticommutes with B_r and thus are the only terms that contribute to the commutator. All these terms are of grade r and the theorem is proved. Additionally, the commutator product obeys the Jacobi identity

$$A\times (B\times C) = (A\times B) \times C + B\times (A\times C) \tag{1.100}$$

This is important for the geometric algebra treatment of Lie groups and algebras.

Chapter 2

Examples of Geometric Algebra

2.1 Quaternions

Any multivector $A \in \mathcal{G}(3,0)$ may be written as

$$A = \alpha + a + B + \beta I \tag{2.1}$$

where $\alpha, \beta \in \Re$, $a \in \mathcal{V}(3,0)$, B is a bivector, and I is the unit pseudoscalar. The quaternions are the multivectors of even grades

$$A = \alpha + B \tag{2.2}$$

B can be represented as

$$B = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} \tag{2.3}$$

where $\mathbf{i} = e_2e_3$, $\mathbf{j} = e_1e_3$, and $\mathbf{k} = e_1e_2$, and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.4}$$

The quaternions form a subalgebra of $\mathcal{G}(3,0)$ since the geometric product of any two quaternions is also a quaternion since the geometric product of two even grade multivector components is a even grade multivector. For example the product of two grade 2 multivectors can only consist of grades 0, 2, and 4, but in $\mathcal{G}(3,0)$ we can only have grades 0 and 2 since the highest possible grade is 3.

2.2 Spinors

The general definition of a spinor is a multivector, $\psi \in \mathcal{G}(p,q)$, such that $\psi v \psi^\dagger \in \mathcal{V}(p,q)\ \ \forall v \in \mathcal{V}(p,q)$. Practically speaking a spinor is the composition of a rotation and a dilation (stretching or shrinking) of a vector. Thus we can write

$$\psi v \psi^\dagger = \rho R v R^\dagger \tag{2.5}$$

where R is a rotor $\left(RR^\dagger = 1\right)$. Letting $U = R^\dagger \psi$ we must solve

$$U v U^\dagger = \rho v \tag{2.6}$$

U must generate a pure dilation. The most general form for U based on the fact that the l.h.s of equation 2.6 must be a vector is

$$U = \alpha + \beta I \tag{2.7}$$

so that

$$U v U^\dagger = \alpha^2 v + \alpha\beta\left(Iv + vI^\dagger\right) + \beta^2 I v I^\dagger = \rho v \tag{2.8}$$

Using $vI^\dagger = (-1)^{\frac{(n-1)(n-2)}{2}} Iv$, $vI^\dagger = (-1)^{n-1} I^\dagger v$, and $II^\dagger = (-1)^q$ we get

$$\alpha^2 v + \alpha\beta\left(1 + (-1)^{\frac{(n-1)(n-2)}{2}}\right) Iv + (-1)^{n+q-1}\beta^2 v = \rho v \tag{2.9}$$

If $\dfrac{(n-1)(n-2)}{2}$ is even $\beta = 0$ and $\alpha \neq 0$, otherwise $\alpha, \beta \neq 0$. For the odd case

$$\psi = R(\alpha + \beta I) \tag{2.10}$$

where $\rho = \alpha^2 + (-1)^{n+q-1}\beta^2$. In the case of $\mathcal{G}(1,3)$ (relativistic space time) we have $\rho = \alpha^2 + \beta^2$, $\rho > 0$.

2.3 Geometric Algebra of the Minkowski Plane

Because of Relativity and QM the Geometric Algebra of the Minkowski Plane is very important for physical applications of Geometric Algebra so we will treat it in detail.

Let the orthonormal basis vectors for the plane be γ_0 and γ_1 where $\gamma_0^2 = -\gamma_1^2 = 1$.[1] Then the geometric product of two vectors $a = a_0\gamma_0 + a_1\gamma_1$ and $b = b_0\gamma_0 + b_1\gamma_1$ is

$$\begin{aligned} ab &= (a_0\gamma_0 + a_1\gamma_1)(b_0\gamma_0 + b_1\gamma_1) && (2.11) \\ &= a_0b_0\gamma_0^2 + a_1b_1\gamma_1^2 + (a_0b_1 - a_1b_0)\gamma_0\gamma_1 && (2.12) \\ &= a_0b_0 - a_1b_1 + (a_0b_1 - a_1b_0) I && (2.13) \end{aligned}$$

so that

$$a \cdot b = a_0b_0 - a_1b_1 \tag{2.14}$$

and

$$a \wedge b = (a_0b_1 - a_1b_0) I \tag{2.15}$$

and

$$I^2 = \gamma_0\gamma_1\gamma_0\gamma_1 = -\gamma_0^2\gamma_1^2 = 1 \tag{2.16}$$

Thus

$$\begin{aligned} e^{\alpha I} &= \sum_{i=0}^{\infty} \frac{\alpha^i I^i}{i!} && (2.17) \\ &= \sum_{i=0}^{\infty} \frac{\alpha^{2i}}{(2i)!} + \sum_{i=0}^{\infty} \frac{\alpha^{2i+1} I}{(2i+1)!} && (2.18) \\ &= \cosh(\alpha) + \sinh(\alpha) I && (2.19) \end{aligned}$$

since $I^{2i} = 1$.

In the Minkowski plane all vectors of the form $a_\pm = \alpha(\gamma_0 \pm \gamma_1)$ are null $\left(a_\pm^2 = 0\right)$. One question to answer are there any vectors $b_\pm$ such that $a_\pm \cdot b_\pm = 0$ that are not parallel to $a_\pm$.

$$\begin{gathered} a_\pm \cdot b_\pm = \alpha\left(b_0^\pm \mp b_1^\pm\right) = 0 \\ b_0^\pm \mp b_1^\pm = 0 \\ b_0^\pm = \pm b_1^\pm \end{gathered}$$

Thus $b_\pm$ must be proportional to $a_\pm$ and the are no vectors in the space that can be constructed that are normal to $a_\pm$. Thus there are no non-zero bivectors, $a \wedge b$, such that $(a \wedge b)^2 = 0$. Conversely, if $a \wedge b \neq 0$ then $(a \wedge b)^2 > 0$.

Finally for the condition that there always exist two orthogonal vectors e_1 and e_2 such that

$$a \wedge b = e_1e_2 \tag{2.20}$$

we can state that neither e_1 nor e_2 can be null.

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[1] $I = \gamma_0\gamma_1$

2.4 Lorentz Transformation

We now have all the tools needed to derive the Lorentz transformation with Geometric Algebra. Consider a two dimensional time-like plane with with coordinates t[2] and x_1 and basis vectors γ_0 and γ_1. Then a general space-time vector in the plane is given by

$$x = t\gamma_0 + x_1\gamma_1 = t'\gamma_0' + x_1'\gamma_1' \tag{2.21}$$

where the basis vectors of the two coordinate systems are related by

$$\gamma_\mu' = R\gamma_\mu R^\dagger \;\; \mu = 0, 1 \tag{2.22}$$

and R is a Minkowski plane rotor

$$R = \sinh\left(\frac{\alpha}{2}\right) + \cosh\left(\frac{\alpha}{2}\right)\gamma_1\gamma_0 \tag{2.23}$$

so that

$$R\gamma_0 R^\dagger = \cosh(\alpha)\,\gamma_0 + \sinh(\alpha)\,\gamma_1 \tag{2.24}$$

and

$$R\gamma_1 R^\dagger = \cosh(\alpha)\,\gamma_1 + \sinh(\alpha)\,\gamma_0 \tag{2.25}$$

Now consider the special case that the primed coordinate system is moving with velocity β in the direction of γ_1 and the two coordinate systems were coincident at time $t = 0$. Then $x_1 = \beta t$ and $x_1' = 0$ so we may write

$$t\gamma_0 + \beta t\gamma_1 = t' R\gamma_0 R^\dagger \tag{2.26}$$

$$\frac{t}{t'}(\gamma_0 + \beta\gamma_1) = \cosh(\alpha)\,\gamma_0 + \sinh(\alpha)\,\gamma_1 \tag{2.27}$$

Equating components gives

$$\cosh(\alpha) = \frac{t}{t'} \tag{2.28}$$

$$\sinh(\alpha) = \frac{t}{t'}\beta \tag{2.29}$$

Solving for α and $\frac{t}{t'}$ in equations 2.28 and 2.29 gives

$$\tanh(\alpha) = \beta \tag{2.30}$$

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[2] We let the speed of light $c = 1$.

$$\frac{t}{t'} = \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{2.31}$$

Now consider the general case of x, t and x', t' giving

$$\begin{aligned} t\gamma_0 + x\gamma_1 &= t' R\gamma_0 R^\dagger + x' R\gamma_1 R^\dagger && (2.32) \\ &= t'\gamma\left(\gamma_0 + \beta\gamma_1\right) + x'\gamma\left(\gamma_1 + \beta\gamma_0\right) && (2.33) \end{aligned}$$

Equating basis vector coefficients recovers the Lorentz transformation

$$\begin{aligned} t &= \gamma\left(t' + \beta x'\right) \\ x &= \gamma\left(x' + \beta t'\right) \end{aligned} \tag{2.34}$$

Chapter 3

Geometric Calculus - The Derivative

3.1 Definitions

If $F(a)$ if a multivector valued function of the vector a, and a and b are any vectors in the space then the derivative of F is defined by

$$b \cdot \nabla F \equiv \lim_{\epsilon \to 0} \frac{F(a + \epsilon b) - F(a)}{\epsilon} \tag{3.1}$$

then letting $b = \boldsymbol{e}_k$ be the components of a coordinate frame with $x = x^k \boldsymbol{e}_k$ (we are using the summation convention that the same upper and lower indices are summed over 1 to N) we have

$$\boldsymbol{e}_k \cdot \nabla F = \lim_{\epsilon \to 0} \frac{F(x^j \boldsymbol{e}_j + \epsilon \boldsymbol{e}_k) - F(x^j \boldsymbol{e}_j)}{\epsilon} \tag{3.2}$$

Using what we know about coordinates gives

$$\nabla F = \boldsymbol{e}^j \frac{\partial F}{\partial x^j} = \boldsymbol{e}^j \partial_j F \tag{3.3}$$

or looking at ∇ as a symbolic operator we may write

$$\nabla = \boldsymbol{e}^j \partial_j \tag{3.4}$$

Due to the properties of coordinate frame expansions ∇F is independent of the choice of the $\boldsymbol{e}_k$ frame. If we consider x to be a position vector then $F(x)$ is in general a multivector field.

3.2 Derivatives of Scalar Functions

If $f(x)$ is scalar valued function of the vector x then the derivative is

$$\nabla f = \boldsymbol{e}^k \partial_k f \tag{3.5}$$

which is the standard definition of the gradient of a scalar function (remember that in an orthonormal coordinate system $\boldsymbol{e}_k = \boldsymbol{e}^k$). Using equation 3.5 we can show the following results for the gradient of some specific scalar functions

$$\begin{array}{rcccc} f & = & x \cdot a, & x^k, & xx \\ \nabla f & = & a, & \boldsymbol{e}^k, & 2x \end{array} \tag{3.6}$$

3.3 Product Rule

Let $\circ$ represent a bilinear product operator such as the geometric, inner, or outer product and note that for the multivector fields F and G we have

$$\partial_k (F \circ G) = (\partial_k F) \circ G + F \circ (\partial_k G) \tag{3.7}$$

so that

$$\begin{aligned} \nabla (F \circ G) &= \boldsymbol{e}^k ((\partial_k F) \circ G + F \circ (\partial_k G)) \\ &= \boldsymbol{e}^k (\partial_k F) \circ G + \boldsymbol{e}^k F \circ (\partial_k G) \end{aligned} \tag{3.8}$$

However since the geometric product is not communicative, in general

$$\nabla (F \circ G) \neq (\nabla F) \circ G + F \circ (\nabla G) \tag{3.9}$$

The notation adopted by Hestenes is

$$\nabla (F \circ G) = \nabla F \circ G + \dot{\nabla} F \circ \dot{G} \tag{3.10}$$

The convention of the overdot notation is

i. In the absence of brackets, ∇ acts on the object to its immediate right

ii. When the ∇ is followed by brackets, the derivative acts on all the the terms in the brackets.

iii. When the ∇ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

Note that with the overdot notation the expression $\dot{A}\dot{\nabla}$ makes sense!

3.4 Interior and Exterior Derivative

The interior and exterior derivatives of an r-grade multivector field are simply defined as (don't forget the summation convention)

$$\nabla \cdot A_r \equiv \left\langle \nabla A_r \right\rangle_{r-1} = \boldsymbol{e}^k \cdot \partial_k A_r \tag{3.11}$$

and

$$\nabla \wedge A_r \equiv \left\langle \nabla A_r \right\rangle_{r+1} = \boldsymbol{e}^k \wedge \partial_k A_r \tag{3.12}$$

Note that

$$\begin{aligned} \nabla \wedge \left(\nabla \wedge A_r\right) &= \boldsymbol{e}^i \partial_i \left(\boldsymbol{e}^j \wedge \partial_j A_r\right) \\ &= \boldsymbol{e}^i \wedge \boldsymbol{e}^j \wedge \left(\partial_i \partial_j A_r\right) \\ &= 0 \end{aligned} \tag{3.13}$$

since $\boldsymbol{e}^i \wedge \boldsymbol{e}^j = -\boldsymbol{e}^j \wedge \boldsymbol{e}^i$, but $\partial_i \partial_j A_r = \partial_j \partial_i A_r$.

$$\begin{aligned} \nabla \cdot \left(\nabla \cdot A_r\right) &= \boldsymbol{e}^i \cdot \partial_i \left(\boldsymbol{e}^j \cdot \partial_j A_r\right) \\ &= \boldsymbol{e}^i \cdot \left(\boldsymbol{e}^j \cdot \left(\partial_i \partial_j A_r\right)\right) \\ &= \pm \boldsymbol{e}^i \cdot \left(\boldsymbol{e}^j \cdot \left(\partial_i \partial_j A_r^* I\right)\right) \\ &= \pm \boldsymbol{e}^i \cdot \left(\left(\boldsymbol{e}^j \wedge \left(\partial_i \partial_j A_r^*\right)\right) I\right) \\ &= \pm \left(\boldsymbol{e}^i \wedge \left(\boldsymbol{e}^j \wedge \left(\partial_i \partial_j A_r^*\right)\right)\right) I \\ &= 0 \end{aligned} \tag{3.14}$$

Where * indicates the dual of a multivector, $A^* = AI$ (I is the pseudoscalar and $A = \pm A^* I$ since $I^2 = \pm 1$), and we use equation 1.53 to exchange the inner and outer products.

Thus for the general multivector field A (built from sums of A_r's) we have $\nabla \wedge (\nabla \wedge A) = 0$ and $\nabla \cdot (\nabla \cdot A) = 0$. If ϕ is a scalar function we also have

$$\begin{aligned} \nabla \wedge (\nabla \phi) &= \boldsymbol{e}^i \wedge \partial_i \left(\boldsymbol{e}^j \partial_j \phi\right) \\ &= \boldsymbol{e}^i \wedge \boldsymbol{e}^j \partial_i \partial_j \phi \\ &= 0 \end{aligned} \tag{3.15}$$

Another use for the overdot notation would in the case where $f\left(x, a\right)$ is a linear function of its second argument ($f(x, \alpha a + \beta b) = \alpha f(x, a) + \beta f(x, b)$) and a is a general function of position

$(a(x) = a^i(x)\,\boldsymbol{e}_i)$. Now calculate

$$\begin{aligned}
\nabla f(x,a) &= \boldsymbol{e}^k \frac{\partial}{\partial x^k} f(x,a) = \boldsymbol{e}^k \frac{\partial}{\partial x^k} f\big(x, a^i(x)\,\boldsymbol{e}_i\big) \tag{3.16}\\
&= \boldsymbol{e}^k \frac{\partial}{\partial x^k}\left(a^i(x)\, f(x,\boldsymbol{e}_i)\right) \tag{3.17}\\
&= \boldsymbol{e}^k \frac{\partial a^i}{\partial x^k} f(x,\boldsymbol{e}_i) + a^i \boldsymbol{e}^k \frac{\partial}{\partial x^k} f(x,\boldsymbol{e}_i) \tag{3.18}\\
&= \boldsymbol{e}^k f\left(x, \frac{\partial a}{\partial x^k}\right) + a^i \boldsymbol{e}^k \frac{\partial}{\partial x^k} f(x,\boldsymbol{e}_i) \tag{3.19}
\end{aligned}$$

Defining

$$\dot{\nabla}\dot{f}\,(a) \equiv a^i \boldsymbol{e}^k \frac{\partial}{\partial x^k} f(x,\boldsymbol{e}_i) = \boldsymbol{e}^k\, \frac{\partial}{\partial x^k} f(x,a)\bigg|_{a=\text{constant}} \tag{3.20}$$

Then suppressing the explicit x dependence of f we get

$$\dot{\nabla}\dot{f}\,(a) = \nabla f\,(a) - \boldsymbol{e}^k f\left(\frac{\partial a}{\partial x^k}\right) \tag{3.21}$$

Other basic results (examples) are

$$\nabla x \cdot A_r = r A_r \tag{3.22}$$

$$\nabla x \wedge A_r = (n-r)\, A_r \tag{3.23}$$

$$\dot{\nabla} A_r \dot{x} = (-1)^r\,(n-2r)\, A_r \tag{3.24}$$

The basic identities for the case of a scalar field α and multivector field F are

$$\nabla\,(\alpha F) = (\nabla\alpha)\, F + \alpha\nabla F \tag{3.25}$$

$$\nabla\cdot(\alpha F) = (\nabla\alpha)\cdot F + \alpha\nabla\cdot F \tag{3.26}$$

$$\nabla\wedge(\alpha F) = (\nabla\alpha)\wedge F + \alpha\nabla\wedge F \tag{3.27}$$

if f_1 and f_2 are vector fields

$$\nabla\wedge(f_1\wedge f_2) = (\nabla\wedge f_1)\wedge f_2 - (\nabla\wedge f_2)\wedge f_1 \tag{3.28}$$

and finally if F_r is a grade r multivector field

$$\nabla\cdot(F_r I) = (\nabla\wedge F_r)\, I \tag{3.29}$$

where I is the psuedoscalar for the geometric algebra.

3.5 Derivative of a Multivector Function

For a vector space of dimension N spanned by the vectors $\boldsymbol{u}_i$ the coordinates of a vector x are the $x^i = x \cdot \boldsymbol{u}^i$ so that $x = x^i \boldsymbol{u}_i$ (summation convention is from 1 to N). Curvilinear coordinates for that space are generated by a one to one invertible differentiable mapping from $\left(x^1, \dots, x^N\right) \leftrightarrow \left(\theta^1, \dots, \theta^N\right)$ where the θ^i are called the curvilinear coordinates. If the mapping is given by $x\left(\theta^1, \dots, \theta^N\right) = x^i\left(\theta^1, \dots, \theta^N\right) \boldsymbol{u}_i$ then the basis vectors associated with the transformation are given by

$$\boldsymbol{e}_k = \frac{\partial x}{\partial \theta^k} = \frac{\partial x^i}{\partial \theta^k} \boldsymbol{u}_i \tag{3.30}$$

The one critical relationship that is required to express the geometric derivative in curvilinear coordinated is

$$\boldsymbol{e}^k = \frac{\partial \theta^k}{\partial x^i} \boldsymbol{u}^i \tag{3.31}$$

The proof is

$$\begin{align} \boldsymbol{e}_j \cdot \boldsymbol{e}^k &= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^n} \boldsymbol{u}_m \cdot \boldsymbol{u}^n \tag{3.32} \\ &= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^n} \delta^n_m \tag{3.33} \\ &= \frac{\partial x^m}{\partial \theta^j} \frac{\partial \theta^k}{\partial x^m} \tag{3.34} \\ &= \frac{\partial \theta^k}{\partial \theta^j} = \delta^k_j \tag{3.35} \end{align}$$

We wish to express the geometric derivative of an R-grade multivector field F_R in terms of the curvilinear coordinates. Thus

$$\nabla F_R = \boldsymbol{u}^i \frac{\partial F_R}{\partial x^i} = \left(\boldsymbol{u}^i \frac{\partial \theta^k}{\partial x^i}\right) \frac{\partial F_R}{\partial \theta^k} = \boldsymbol{e}^k \frac{\partial F_R}{\partial \theta^k} \tag{3.36}$$

Note that if we start by defining the $\boldsymbol{e}_k$'s the reciprocal frame vectors $\boldsymbol{e}^k$ can be calculated geometrically (we do not need the inverse partial derivatives). Now define a new blade symbol by

$$\boldsymbol{e}_{[i_1,\dots,i_R]} = \boldsymbol{e}_{i_1} \wedge \dots \wedge \boldsymbol{e}_{i_R} \tag{3.37}$$

and represent an R-grade multivector function F by

$$F = F^{i_1 \dots i_R} \boldsymbol{e}_{[i_1,\dots,i_R]} \tag{3.38}$$

Then

$$\nabla F = \frac{\partial F^{i_1\ldots i_R}}{\partial \theta^k}\boldsymbol{e}^k\boldsymbol{e}_{[i_1,\ldots,i_R]} + F^{i_1\ldots i_R}\boldsymbol{e}^k\frac{\partial}{\partial \theta^k}\boldsymbol{e}_{[i_1,\ldots,i_R]} \tag{3.39}$$

Define

$$C\left\{\boldsymbol{e}_{[i_1,\ldots,i_R]}\right\} \equiv \boldsymbol{e}^k\frac{\partial}{\partial \theta^k}\boldsymbol{e}_{[i_1,\ldots,i_R]} \tag{3.40}$$

Where $C\left\{\boldsymbol{e}_{[i_1,\ldots,i_R]}\right\}$ are the connection multivectors for each base of the geometric algebra and we can write

$$\nabla F = \frac{\partial F^{i_1\ldots i_R}}{\partial \theta^k}\boldsymbol{e}^k\boldsymbol{e}_{[i_1,\ldots,i_R]} + F^{i_1\ldots i_R}C\left\{\boldsymbol{e}_{[i_1,\ldots,i_R]}\right\} \tag{3.41}$$

Note that all the quantities in the equation not dependent upon the $F^{i_1\ldots i_R}$ can be directly calculated if the $\boldsymbol{e}_k\left(\theta^1,\ldots,\theta^N\right)$ is known so further simplification is not needed.

In general the $\boldsymbol{e}_k$'s we have defined are not normalized so define

$$|\boldsymbol{e}_k| = \sqrt{|\boldsymbol{e}_k^2|} \tag{3.42}$$

$$\hat{\boldsymbol{e}}_k = \frac{\boldsymbol{e}_k}{|\boldsymbol{e}_k|} \tag{3.43}$$

and note that $\hat{\boldsymbol{e}}_k^2 = \pm 1$ depending upon the metric. Note also that

$$\hat{\boldsymbol{e}}^k = |\boldsymbol{e}_k|\,\boldsymbol{e}^k \tag{3.44}$$

since

$$\hat{\boldsymbol{e}}^j \cdot \hat{\boldsymbol{e}}_k = \left(|\boldsymbol{e}_j|\,\boldsymbol{e}^j\right)\cdot\left(\frac{\boldsymbol{e}_k}{|\boldsymbol{e}_k|}\right) = \delta^j_k\frac{|\boldsymbol{e}_j|}{|\boldsymbol{e}_k|} = \delta^j_k \tag{3.45}$$

so that if F_R is represented in terms of the normalized basis vectors we have

$$F_R = F_R^{i_1\ldots i_R}\hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \tag{3.46}$$

and the geometric derivative is now

$$\nabla F = \frac{\partial F^{i_1\ldots i_R}}{\partial \theta^k}\frac{\hat{\boldsymbol{e}}^k}{|\boldsymbol{e}_k|}\hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} + F^{i_1\ldots i_R}\hat{C}\left\{\hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]}\right\} \tag{3.47}$$

and

$$\hat{C}\left\{\hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]}\right\} = \frac{\hat{\boldsymbol{e}}^k}{|\boldsymbol{e}_k|}\frac{\partial}{\partial \theta^k}\hat{\boldsymbol{e}}_{[i_1,\ldots,i_R]} \tag{3.48}$$

3.5.1 Spherical Coordinates

For spherical coordinates the coordinate generating function is:

$$x = r\left(\cos\left(\theta\right)\boldsymbol{u}_{z} + \sin\left(\theta\right)\left(\cos\left(\phi\right)\boldsymbol{u}_{x} + \sin\left(\phi\right)\boldsymbol{u}_{y}\right)\right) \tag{3.49}$$

so that

$$\boldsymbol{e}_{r} = \cos\left(\theta\right)\left(\cos\left(\phi\right)\boldsymbol{u}_{x} + \sin\left(\phi\right)\boldsymbol{u}_{y}\right) + \sin\left(\theta\right)\boldsymbol{u}_{z} \tag{3.50}$$

$$\boldsymbol{e}_{\theta} = r\left(-\sin\left(\theta\right)\left(\cos\left(\phi\right)\boldsymbol{u}_{x} + \sin\left(\phi\right)\boldsymbol{u}_{y}\right) + \cos\left(\theta\right)\boldsymbol{u}_{z}\right) \tag{3.51}$$

$$\boldsymbol{e}_{\phi} = r\cos\left(\theta\right)\left(-\sin\left(\phi\right)\boldsymbol{u}_{x} + \cos\left(\phi\right)\boldsymbol{u}_{y}\right) \tag{3.52}$$

where

$$\left|\boldsymbol{e}_{r}\right| = 1 \quad \left|\boldsymbol{e}_{\theta}\right| = r \quad \left|\boldsymbol{e}_{\phi}\right| = r\sin\left(\theta\right) \tag{3.53}$$

and

$$\boldsymbol{\hat{e}}_{r} = \cos\left(\theta\right)\left(\cos\left(\phi\right)\boldsymbol{u}_{x} + \sin\left(\phi\right)\boldsymbol{u}_{y}\right) + \sin\left(\theta\right)\boldsymbol{u}_{z} \tag{3.54}$$

$$\boldsymbol{\hat{e}}_{\theta} = -\sin\left(\theta\right)\left(\cos\left(\phi\right)\boldsymbol{u}_{x} + \sin\left(\phi\right)\boldsymbol{u}_{y}\right) + \cos\left(\theta\right)\boldsymbol{u}_{z} \tag{3.55}$$

$$\boldsymbol{\hat{e}}_{\phi} = -\sin\left(\phi\right)\boldsymbol{u}_{x} + \cos\left(\phi\right)\boldsymbol{u}_{y} \tag{3.56}$$

the connection mulitvectors for the normalize basis vectors are

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{r}\right\} = \frac{2}{r} \tag{3.57}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{\theta}\right\} = \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)} + \frac{1}{r}\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\theta} \tag{3.58}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{\phi}\right\} = \frac{1}{r}\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\phi} + \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\boldsymbol{\hat{e}}_{\theta}\wedge\boldsymbol{\hat{e}}_{\phi} \tag{3.59}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\theta}\right\} = -\frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\boldsymbol{\hat{e}}_{r} + \frac{1}{r}\boldsymbol{\hat{e}}_{\theta} \tag{3.60}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\phi}\right\} = \frac{1}{r}\boldsymbol{\hat{e}}_{\phi} - \frac{\cos\left(\theta\right)}{r\sin\left(\theta\right)}\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\theta}\wedge\boldsymbol{\hat{e}}_{\phi} \tag{3.61}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{\theta}\wedge\boldsymbol{\hat{e}}_{\phi}\right\} = \frac{2}{r}\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\theta}\wedge\boldsymbol{\hat{e}}_{\phi} \tag{3.62}$$

$$\hat{C}\left\{\boldsymbol{\hat{e}}_{r}\wedge\boldsymbol{\hat{e}}_{\theta}\wedge\boldsymbol{\hat{e}}_{\phi}\right\} = 0 \tag{3.63}$$

For a vector function A using equation 3.41 and that $\nabla A = \nabla\cdot A + \nabla\wedge A$

$$\begin{aligned}\nabla\cdot A &= \frac{1}{r\sin(\theta)}\left(A^{\theta}\cos(\theta)+\partial_{\phi}A^{\phi}\right)+\frac{1}{r}\left(2A^{r}+\partial_{\theta}A^{\theta}\right)+\partial_{r}A^{r} &(3.64)\\ &= \frac{1}{r^{2}}\partial_{r}\left(r^{2}A^{r}\right)+\frac{1}{r\sin(\theta)}\left(\partial_{\theta}\left(\sin(\theta)A^{\theta}\right)+\partial_{\phi}A^{\phi}\right) &(3.65)\end{aligned}$$

$$\begin{aligned}\nabla\times A &= -I\left(\nabla\wedge A\right) &(3.66)\\ &= \left(\frac{\partial_{\theta}A^{\phi}}{r}+\frac{1}{r\sin(\theta)}\left(A^{\phi}\cos(\theta)-\partial_{\phi}A^{\theta}\right)\right)\boldsymbol{\hat{e}}_{r} &(3.67)\\ &\quad+\left(\frac{\partial_{\phi}A^{r}}{r\sin(\theta)}-\frac{A^{\phi}}{r}-\partial_{r}A^{\phi}\right)\boldsymbol{\hat{e}}_{\theta} &(3.68)\\ &\quad+\left(\frac{A^{\theta}}{r}+\partial_{r}A^{\theta}-\frac{\partial_{\theta}A^{r}}{r}\right)\boldsymbol{\hat{e}}_{\phi} &(3.69)\end{aligned}$$

$$\begin{aligned}\nabla\times A =&\frac{1}{r\sin(\theta)}\left(\partial_{\theta}\left(\sin(\theta)A^{\phi}\right)-\partial_{\phi}A^{\theta}\right)\boldsymbol{\hat{e}}_{r} &(3.70)\\ &+\frac{1}{r}\left(\frac{1}{\sin(\theta)}\partial_{\phi}A^{r}-\partial_{r}\left(rA^{\phi}\right)\right)\boldsymbol{\hat{e}}_{\theta} &(3.71)\\ &+\frac{1}{r}\left(\partial_{r}\left(rA^{\theta}\right)-\partial_{\theta}A^{r}\right)\boldsymbol{\hat{e}}_{\phi} &(3.72)\end{aligned}$$

These are the standard formulas for div and curl in spherical coordinates.

3.6 Analytic Functions

Starting with $\mathcal{G}(2,0)$ and orthonormal basis vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ so that $I=\boldsymbol{e}_x\boldsymbol{e}_y$ and $I^2=-1$. Then we have

$$\mathbf{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y \tag{3.73}$$

$$\nabla = \boldsymbol{e}_x\frac{\partial}{\partial x}+\boldsymbol{e}_y\frac{\partial}{\partial y} \tag{3.74}$$

Map $\mathbf{r}$ onto the complex number z via

$$z = x + Iy = \boldsymbol{e}_x\mathbf{r} \tag{3.75}$$

Define the multivector field $\psi = u + Iv$ where u and v are scalar fields. Then

$$\nabla\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\boldsymbol{e}_x + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\boldsymbol{e}_y \tag{3.76}$$

Thus the statement that ψ is an analytic function is equivalent to

$$\nabla\psi = 0 \tag{3.77}$$

This is the fundamental equation that can be generalized to higher dimensions remembering that in general that ψ is a multivector rather than a scalar function! To complete the connection with complex analysis we define ($z^\dagger = x - Iy$)

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) \tag{3.78}$$

so that

$$\begin{array}{ll} \dfrac{\partial z}{\partial z} = 1, & \dfrac{\partial z^\dagger}{\partial z} = 0 \\ \dfrac{\partial z}{\partial z^\dagger} = 0, & \dfrac{\partial z^\dagger}{\partial z^\dagger} = 1 \end{array} \tag{3.79}$$

An analytic function is one that depends on z alone so that we can write $\psi(x + Iy) = \psi(z)$ and

$$\frac{\partial\psi(z)}{\partial z^\dagger} = 0 \tag{3.80}$$

equivalently

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)\psi = \frac{1}{2}\boldsymbol{e}_x\nabla\psi = 0 \tag{3.81}$$

Now it is simple to show why solutions to $\nabla\psi = 0$ can be written as a power series in z. First

$$\begin{aligned} \nabla z &= \nabla(\boldsymbol{e}_x\mathbf{r}) \\ &= \boldsymbol{e}_x\boldsymbol{e}_x\frac{\partial\mathbf{r}}{\partial x} + \boldsymbol{e}_y\boldsymbol{e}_x\frac{\partial\mathbf{r}}{\partial y} \\ &= \boldsymbol{e}_x\boldsymbol{e}_x\boldsymbol{e}_x + \boldsymbol{e}_y\boldsymbol{e}_x\boldsymbol{e}_y \\ &= \boldsymbol{e}_x - \boldsymbol{e}_x \\ &= 0 \end{aligned} \tag{3.82}$$

so that

$$\nabla(z - z_0)^k = k\nabla(\boldsymbol{e}_x\mathbf{r} - z_0)(z - z_0)^{k-1} = 0 \tag{3.83}$$

Chapter 4

Geometric Calculus - Integration

4.1 Line Integrals

If $F(x)$ is a multivector field and $x(\lambda)$ is a parametric representation of a vector path (curve) then the line Integral of F along the path x is defined to be

$$\int F(x)\frac{dx}{d\lambda}d\lambda = \int F\,dx \equiv \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta x^i \tag{4.1}$$

where

$$\Delta x^i = x_i - x_{i-1}, \quad \bar{F}^i = \frac{1}{2}\left(F(x_{i-1}) + F(x_i)\right) \tag{4.2}$$

if $x_n = x_1$ the path is closed. Since dx is a vector, that is $F(x)\frac{dx}{d\lambda} \neq \frac{dx}{d\lambda}F(x)$, a more general line integral would be

$$\int F(x)\frac{dx}{d\lambda}G(x)\; d\lambda = \int F(x)\,dx\,G(x) \tag{4.3}$$

The most general form of line integral would be

$$\int \mathsf{L}(\partial_\lambda x; x)\; d\lambda = \int \mathsf{L}(dx) \tag{4.4}$$

where $\mathsf{L}(a) = \mathsf{L}(a;x) =$ is a multivector-valued linear function of a. The position dependence in L can often be suppressed to streamline the notation.

4.2 Surface Integrals

The next step is a directed surface integral. Let $F(x)$ be a multivector field and let a surface be parametrized by two coordinates $x(x^1, x^2)$. Then we can define a directed surface measure by

$$dX = \frac{\partial x}{\partial x^1} \wedge \frac{\partial x}{\partial x^2}\, dx^1 dx^2 = \boldsymbol{e}_1 \wedge \boldsymbol{e}_2\, dx^1 dx^2 \tag{4.5}$$

A directed surface integral takes the form

$$\int F\, dX = \int F \boldsymbol{e}_1 \wedge \boldsymbol{e}_2\, dx^1 dx^2 \tag{4.6}$$

In order to construct some of the more important proof it is necessary to express the surface integral as the limit of a sum. This requires the concept of a triangulated surface as shown Each triangle in the surface is described by a planar simplex as shown The three vertices of the planar simplex are x_0, x_1, and x_2 with the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ defined by

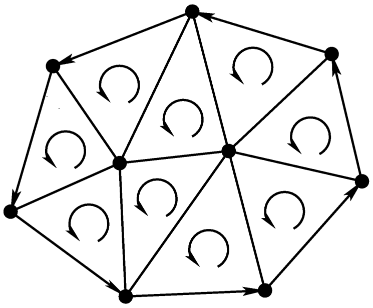

Figure 4.1: Triangulated Surface

$$\boldsymbol{e}_1 = x_1 - x_0, \quad \boldsymbol{e}_2 = x_2 - x_0 \tag{4.7}$$

so that the surface measure of the simplex is

$$\Delta X \equiv \frac{1}{2}\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 = \frac{1}{2}\left(x_1 \wedge x_2 + x_2 \wedge x_0 + x_0 \wedge x_1\right) \tag{4.8}$$

with this definition of ΔX we have

$$\int F\, dX \equiv \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{F}^k \Delta X^k \tag{4.9}$$

where $\bar{F}^k$ is the average of F over the k^{th} simplex.

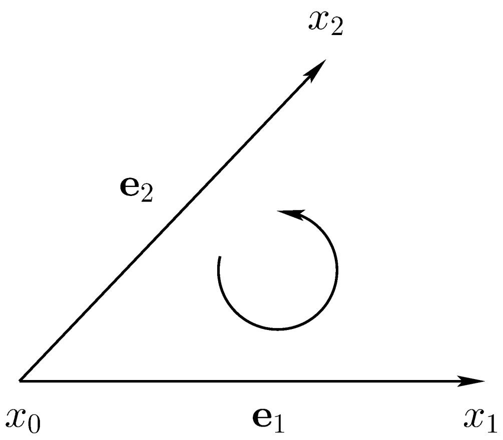

Figure 4.2: Planar Simplex

4.3 Directed Integration - n-dimensional Surfaces

4.3.1 k-Simplex Definition

In geometry, a simplex or k-simplex is an k-dimensional analogue of a triangle. Specifically, a simplex is the convex hull of a set of $(k+1)$ affinely independent points in some Euclidean space of dimension k or higher.

For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron (in each case including the interior). A regular simplex is a simplex that is also a regular polytope. A regular k-simplex may be constructed from a regular $(k-1)$-simplex by connecting a new vertex to all original vertices by the common edge length.

4.3.2 k-Chain Definition (Algebraic Topology)

A finite set of k-simplexes embedded in an open subset of $\Re^n$ is called an affine k-chain. The simplexes in a chain need not be unique, they may occur with multiplicity. Rather than using standard set notation to denote an affine chain, the standard practice to use plus signs to separate each member in the set. If some of the simplexes have the opposite orientation, these are prefixed by a minus sign. If some of the simplexes occur in the set more than once, these are prefixed with an integer count. Thus, an affine chain takes the symbolic form of a sum with integer coefficients.

4.3.3 Simplex Notation

If $(x_0, x_1, \ldots, x_k)$ is the k-simplex defined by the $k+1$ points $x_0, x_1, \ldots, x_k$. This is abbreviated by

$$(x)_{(k)} = (x_0, x_1, \ldots, x_k) \tag{4.10}$$

The order of the points is important for a simplex, since it specifies the orientation of the simplex. If any two adjacent points are swapped the simplex orientation changes sign. The boundary operator for the simplex is denoted by ∂ and defined by

$$\partial\,(x)_{(k)} \equiv \sum_{i=0}^{k} (-1)^i \, (x_0, \ldots, \breve{x}_i, \ldots, x_k)_{(k-1)} \tag{4.11}$$

To see that this make sense consider a triangle $(x)_{(3)} = (x_0, x_1, x_2)$. Then

$$\begin{aligned}\partial\,(x)_{(3)} &= (x_1, x_2)_{(2)} - (x_0, x_2)_{(2)} + (x_0, x_1)_{(2)} \\ &= (x_1, x_2)_{(2)} + (x_2, x_0)_{(2)} + (x_0, x_1)_{(2)}\end{aligned} \tag{4.12}$$

each 2-simplex in the boundary 2-chain connects head to tail with the same sign.

Now consider the boundary of the boundary

$$\begin{aligned}\partial^2\,(x)_{(3)} &= \partial\,(x_1, x_2)_{(2)} + \partial\,(x_2, x_0)_{(2)} + \partial\,(x_0, x_1)_{(2)} \\ &= (x_1)_{(1)} - (x_2)_{(1)} + (x_2)_{(1)} - (x_0)_{(1)} + (x_0)_{(1)} - (x_1)_{(1)} \\ &= 0\end{aligned} \tag{4.13}$$

We need to prove is that in general $\partial^2\,(x)_{(k)} = 0$. To do this consider the boundary of the i^{th} term on thr r.h.s. of equation 4.11 letting $A_{ij}^{(k-2)} = (x_0, \ldots, \breve{x}_i, \ldots, \breve{x}_j, \ldots, x_k)_{(k-1)}$.

Then

$$\partial\,(x_0, \ldots, \breve{x}_i, \ldots, x_k)_{(k-1)} = \left\{\begin{array}{cc} i = 0: & \displaystyle\sum_{j=1}^{k} (-1)^{j-1} \, A_{ij}^{(k-2)} \\ 0 < i < k: & \displaystyle\sum_{j=0}^{i-1} (-1)^j \, A_{ij}^{(k-2)} + \sum_{j=i+1}^{k} (-1)^{j-1} \, A_{ij}^{(k-2)} \\ i = k: & \displaystyle\sum_{j=0}^{k-1} (-1)^j \, A_{ij}^{(k-2)} \end{array}\right\} \tag{4.14}$$

The critical point in equation 4.14 is that the exponent of -1 in the second term on the r.h.s. is not j, but $j-1$. The reason for this is that when x_i was removed from the simplex the vertices were **not** renumbered. We can now express the boundary of the boundary in terms of the following matrix elements ($B_{ij}^{(k-2)} = (-1)^{i+j} A_{ij}^{(k-2)}$) as

$$\begin{aligned}
\partial^2 (x)_{(k)} =& \sum_{j=1}^{k} (-1)^{j-1} A_{0j}^{(k-2)} + (-1)^k \sum_{j=0}^{k-1} (-1)^j A_{kj}^{(k-2)} \\
&+ \sum_{i=1}^{k-1} (-1)^i \left(\sum_{j=0}^{i-1} (-1)^j A_{ij}^{(k-2)} + \sum_{j=i+1}^{k} (-1)^{j-1} A_{ij}^{(k-2)} \right) \\
=& -\sum_{j=1}^{k} B_{0j}^{(k-2)} + \sum_{j=0}^{k-1} B_{kj}^{(k-2)} \\
&+ \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} B_{ij}^{(k-2)} - \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} B_{ij}^{(k-2)} = 0
\end{aligned} \tag{4.15}$$

The consider $B_{ij}^{(k-2)}$ as a matrix (i-row index, j-column index). The matrix is symmetrical and in equation 4.15 you are subtracting all the elements above the main diagonal from the elements below the main diagonal so that $\partial^2 (x)_{(k)} = 0$ and the boundary of a boundary of a simplex is zero.

Now add geometry to the simplex by defining the vectors

$$e_i = x_i - x_0, \quad i = 1, \ldots, k, \tag{4.16}$$

and the directed volume element

$$\Delta X = \frac{1}{k!} e_1 \wedge \cdots \wedge e_k \tag{4.17}$$

We now wish to prove that

$$\int_{(x)_{(k)}} dX = \Delta X \tag{4.18}$$

Any point in the simplex can be written in terms of the coordinates λ^i as

$$x = x_0 + \sum_{i=1}^{k} \lambda^i e_i \tag{4.19}$$

with restrictions

$$0 \le \lambda^i \le 1 \quad \text{and} \quad \sum_{i=1}^{k} \lambda^i \le 1 \tag{4.20}$$

First we show that

$$\int_{(x)_{(k)}} dX = \int_{(x)_{(k)}} e_1 \wedge \cdots \wedge e_k \, d\lambda^1 \cdots d\lambda^k = \Delta X \tag{4.21}$$

or

$$\int_{(x)_{(k)}} d\lambda^1 \cdots d\lambda^k = \frac{1}{k!} \tag{4.22}$$

define $\Lambda_j = 1 - \sum_{i=1}^{j} \lambda^i$ (Note that $\Lambda_0 = 1$). From the restrictions on the λ^i's we have

$$\int_{(x)_{(k)}} d\lambda^1 \cdots d\lambda^k = \int_0^{\Lambda_0} d\lambda^1 \int_0^{\Lambda_1} d\lambda^2 \cdots \int_0^{\Lambda_{k-1}} d\lambda^k \tag{4.23}$$

To prove that the r.h.s of equation 4.23 is $1/k!$ we form the following sequence and use induction to prove that V_j is the result of the first j partial Integrations of equation 4.23

$$V_j = \frac{1}{j!} \left(\Lambda_{k-j}\right)^j \tag{4.24}$$

Then

$$\begin{aligned} V_{j+1} &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j} V_j \\ &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j} \frac{1}{j!} \left(\Lambda_{k-j-1} - \lambda^{k-j}\right)^j \\ &= \frac{-1}{(j+1)\, j!} \left[\left(\Lambda_{k-j-1} - \lambda^{k-j}\right)^{j+1}\right]_0^{\Lambda_{k-j-1}} \\ &= \frac{1}{(j+1)!} \left(\Lambda_{k-j-1}\right)^{j+1} \end{aligned} \tag{4.25}$$

so that $V_k = 1/k!$ and the assertion is proved. Now let there be a multivector field $F(x)$ that assumes the values $F_i = F(x_i)$ at the vertices of the simplex and define the interpolating function

$$f(x) = F_0 + \sum_{i=1}^{k} \lambda^i \left(F_i - F_0\right) \tag{4.26}$$

We now wish to show that

$$\int_{(x)_{(k)}} f\, dX = \frac{1}{k+1}\left(\sum_{i=0}^{k} F_i\right)\Delta X = \bar{F}\,\Delta X \tag{4.27}$$

To prove this we must show that

$$\int_{(x)_{(k)}} \lambda^i\, dX = \frac{1}{k+1}\Delta X, \quad \forall\, \lambda^i \tag{4.28}$$

To do this consider the integral (equation 4.25 with V_j replaced by $\lambda^{k-j}V_j$)

$$\begin{aligned}\int_0^{\Lambda_{k-j-1}} d\lambda^{k-j}\, \lambda^{k-j}V_j &= \int_0^{\Lambda_{k-j-1}} d\lambda^{k-j}\frac{1}{j!}\lambda^{k-j}\left(\Lambda_{k-j-1}-\lambda^{k-j}\right)^j \\ &= \frac{1}{(j+2)!}\left(\Lambda_{k-j-1}\right)^{j+2} \end{aligned}\tag{4.29}$$

Note that since the extra λ^i factor occurs in exactly one of the subintegrals for each different λ^i the final result of the total integral is multiplied by a factor of $\frac{1}{(k+1)}$ since the weight of the total integral is now $\frac{1}{(k+1)!}$ and the assertion (equation 4.28 and hence equation 4.27) is proved.

Now summing over all the simplices making up the directed volume gives

$$\int_{\text{volume}} F\, dX = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta X^i \tag{4.30}$$

The most general statement of equation 4.30 is

$$\int_{\text{volume}} \mathsf{L}\left(dX\right) = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{\mathsf{L}}^i\left(\Delta X^i\right) \tag{4.31}$$

where $\mathsf{L}\left(F_n; x\right)$ is a position dependent linear function of a grade-n multivector F_n and $\bar{\mathsf{L}}^i$ is the average value of $\mathsf{L}\left(dX\right)$ over the vertices of each simplex.

An example of this would be

$$\mathsf{L}\left(F_n; x\right) = G\left(x\right)F_n H\left(x\right) \tag{4.32}$$

where $G\left(x\right)$ and $H\left(x\right)$ are multivector functions of x.

4.4 Fundamental Theorem of Geometric Calculus

Now prove that the directed measure of a simplex boundary is zero

$$\Delta\left(\partial\left(x\right)_{(k)}\right)=\Delta\left(\partial\left(x_0,\dots,x_k\right)_{(k)}\right)=0 \tag{4.33}$$

Start with a planar simplex of three points

$$\partial\left(x_0,x_1,x_2\right)_{(2)}=\left(x_1,x_2\right)_{(1)}-\left(x_0,x_2\right)_{(1)}+\left(x_0,x_1\right)_{(1)} \tag{4.34}$$

so that

$$\Delta\left(\partial\left(x_0,x_1,x_2\right)_{(2)}\right)=\left(x_2-x_1\right)-\left(x_2-x_0\right)+\left(x_1-x_0\right)=0 \tag{4.35}$$

We shall now prove equation 4.33 via induction. First note that

$$\Delta\left(\breve{x}_i\right)_{(k-1)}=\left\{\begin{array}{cc} i=0: & \dfrac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\left(x_k-x_1\right) \\ 0<i\le k-1: & \dfrac{1}{k-1}\Delta\left(\breve{x}_i\right)_{(k-2)}\wedge\left(x_k-x_0\right)\end{array}\right\} \tag{4.36}$$

and

$$\Delta\left(\breve{x}_k\right)_{(k-1)}=\frac{1}{(k-1)!}\left(x_1-x_0\right)\wedge\cdots\wedge\left(x_{k-1}-x_0\right) \tag{4.37}$$

so that

$$\Delta\left(\partial\left(x\right)_{(k)}\right)=\frac{1}{k-1}\sum_{i=1}^{k-1}\left(-1\right)^i\Delta\left(\breve{x}_i\right)_{(k-2)}\wedge\left(x_k-x_0\right)+\mathcal{C}$$

where

$$\mathcal{C}=\frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\left(x_k-x_1\right)+\left(-1\right)^k\Delta\left(\breve{x}_k\right)_{(k-1)} \tag{4.38}$$

if we let $\delta=x_0-x_1$ we can write

$$\mathcal{C}=\frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\left(x_k-x_0\right)+\frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\delta+\left(-1\right)^k\Delta\left(\breve{x}_k\right)_{(k-1)} \tag{4.39}$$

Then

$$\begin{aligned}
\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\delta &= \frac{1}{(k-2)!}\left(x_2-x_1\right)\wedge\cdots\wedge\left(x_{k-1}-x_1\right)\wedge\delta && (4.40)\\
&= \frac{1}{(k-2)!}\left(x_2-x_0+\delta\right)\wedge\cdots\wedge\left(x_{k-1}-x_0+\delta\right)\wedge\delta && (4.41)\\
&= \frac{1}{(k-2)!}\left(x_2-x_0\right)\wedge\cdots\wedge\left(x_{k-1}-x_0\right)\wedge\delta && (4.42)\\
&= \frac{(-1)^{k-2}}{(k-2)!}\delta\wedge\left(x_2-x_0\right)\wedge\cdots\wedge\left(x_{k-1}-x_0\right) && (4.43)\\
&= \frac{(-1)^{k-1}}{(k-2)!}\left(x_1-x_0\right)\wedge\left(x_2-x_0\right)\wedge\cdots\wedge\left(x_{k-1}-x_0\right) && (4.44)
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{-1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\delta = && (4.45)\\
&= \frac{(-1)^k}{(k-1)!}\left(x_1-x_0\right)\wedge\left(x_2-x_0\right)\wedge\cdots\wedge\left(x_{k-1}-x_0\right) && (4.46)\\
&= (-1)^k\,\Delta\left(\breve{x}_k\right)_{(k-1)} && (4.47)
\end{aligned}$$

However

$$(-1)^k\,\Delta\left(\breve{x}_k\right)_{(k-1)} = \frac{-1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\delta \tag{4.48}$$

so that

$$\mathcal{C} = \frac{1}{k-1}\Delta\left(\breve{x}_0\right)_{(k-2)}\wedge\left(x_k-x_0\right) \tag{4.49}$$

and

$$\begin{aligned}
\Delta\left(\partial\left(x\right)_{(k)}\right) &= \frac{1}{k-1}\left(\sum_{i=0}^{k-1}(-1)^i\,\Delta\left(\breve{x}_i\right)_{(k-2)}\right)\wedge\left(x_k-x_0\right)\\
&= \frac{1}{k-1}\left(\Delta\left(\partial\left(x\right)_{(k-1)}\right)\right)\wedge\left(x_k-x_0\right)\\
&= 0 && (4.50)
\end{aligned}$$

We have proved equation 4.33. Note that to reduce equation 4.49 we had to use that for any set of vectors $\delta, y_1, \ldots, y_k$ we have

$$\delta\wedge\left(y_1+\delta\right)\wedge\cdots\wedge\left(y_k+\delta\right) = \delta\wedge y_1\wedge\cdots\wedge y_k \tag{4.51}$$

Think about equation 4.51. It's easy to prove $(\delta\wedge\delta=0)$!

Equation 4.33 is sufficient to prove that the directed integral over the surface of simplex is zero

$$\oint_{\partial(x)_{(k)}} dS = \sum_{i=0}^{k}(-1)^i \int_{(\check{x}_i)_{(k-1)}} dX = \Delta\left(\partial\,(x)_{(k)}\right) = 0 \tag{4.52}$$

The characteristics of a general volume are:

1. A general volume is built up from a chain of simplices.
2. Simplices in the chain are defined so that at any common boundary the directed areas of the bounding faces are equal and opposite.
3. Surface integrals over two simplices cancel over their common face.
4. The surface integral over the boundary of the volume can be replaced by the sum of the surface integrals over each simplex in the chain.

If the boundary of the volume is closed we have

$$\oint dS = \lim_{n\mapsto\infty}\sum_{a=1}^{n}\oint dS^a = 0 \tag{4.53}$$

Where $\oint dS^a$ is the surface Integral over the a^{th} simplex. Implicit in equation 4.53 is that the surface is orientated, simply connected, and closed.

The next lemma to prove that if b is a constant vector on the simplex $(x)_{(k)}$ then

$$\oint_{\partial(x)_{(k)}} b\cdot x\; dS = b\cdot\Delta\left((x)_{(k)}\right) = b\cdot\Delta X \tag{4.54}$$

The starting point of the lemma is equation 4.28. First define

$$b = \sum_{i=1}^{k} b_i e^i, \tag{4.55}$$

where the e^i's are the reciprocal frame to $e_i = x_i - x_0$ so that

$$x - x_0 = \sum_{i=1}^{k}\lambda^i e_i, \tag{4.56}$$

$$b_i = b \cdot e_i, \tag{4.57}$$

and

$$\sum_{i=1}^{k} \lambda^i b_i = b \cdot (x - x_0)\,. \tag{4.58}$$

Substituting into equation 4.28 we get

$$\sum_{i=1}^{k} \int_{(x)_{(k)}} b_i \lambda^i \; dX = \int_{(x)_{(k)}} b \cdot (x - x_0) \; dX = \frac{1}{k+1} \sum_{i=1}^{k} b \cdot e_i \; \Delta X \tag{4.59}$$

Rearranging terms gives

$$\begin{aligned} \int_{(x)_{(k)}} b \cdot x \; dX &= \frac{1}{k+1} \left(\left(\sum_{i=1}^{k} b \cdot (x_i - x_0) \right) + (k+1)\, b \cdot x_0 \right) \Delta X \\ &= \frac{b}{k+1} \cdot \left(\sum_{i=0}^{k} x_i \right) \Delta X \\ &= b \cdot \bar{x} \Delta X \end{aligned} \tag{4.60}$$

Now using the definition of a simplex boundary (equation 4.11) and equation 4.60 we get

$$\oint_{\partial (x)_{(k)}} b \cdot x \; dS = \frac{1}{k} \sum_{i=0}^{k} (-1)^i \, b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k)\, \Delta \left((\breve{x}_i)_{(k-1)} \right) \tag{4.61}$$

The coefficient multiplying the r.h.s. of equation 4.61 is $\frac{1}{k}$ and not $\frac{1}{k+1}$ because both $(x_0 + \cdots + \breve{x}_i + \cdots + x_k)$ and $(\breve{x}_i)_{(k-1)}$ refer to $k-1$ simplices (the boundary of $(x)_{(k)}$ is the sum of all the simplices $(\breve{x}_i)_{(k)}$ with proper sign assigned).

Now to prove equation 4.54 we need to prove one final purely algebraic lemma

$$\sum_{i=0}^{k} (-1)^i \, b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k)\, \Delta\, (\breve{x}_i)_{(k-1)} = \frac{1}{(k-1)!} b \cdot (e_1 \wedge \cdots \wedge e_k) \tag{4.62}$$

Begin with the definition of the l.h.s. of equation 4.62

$$\mathsf{C} = \sum_{i=0}^{k} (-1)^i \, b \cdot (x_0 + \cdots + \breve{x}_i + \cdots + x_k)\, \Delta\, (\breve{x}_i)_{(k-1)} = \sum_{i=0}^{k} \mathsf{C}_i \tag{4.63}$$

where C_i is defined by

$$\mathsf{C}_i = \left\{ \begin{array}{cc} i=0: & b\cdot(x_1+\cdots+x_k)\,\Delta\,(\breve{x}_0)_{(k-1)} \\ 0<i\leq k: & (-1)^i\,b\cdot(x_0+\cdots+\breve{x}_i+\cdots+x_k)\,\Delta\,(\breve{x}_i)_{(k-1)} \end{array} \right\} \tag{4.64}$$

now define $E_k = e_1\wedge\cdots\wedge e_k$ so that (using equation 1.61 from the section on reciprocal frames)

$$(-1)^{i-1}\,e^iE_k = e_1\wedge\cdots\wedge\breve{e}_i\wedge\cdots\wedge e_k, \quad \forall\, 0<i\leq k \tag{4.65}$$

and

$$\mathsf{C}_{0<i\leq k} = \frac{-1}{(k-1)!}b\cdot(x_0+\cdots+\breve{x}_i+\cdots+x_k)\,e^iE_k \tag{4.66}$$

The main problem is in evaluating C_0 since

$$\Delta\,(\breve{x}_0)_{(k-1)} = \frac{1}{(k-1)!}\,(x_2-x_1)\wedge\cdots\wedge(x_k-x_1) \tag{4.67}$$

using $e_i = x_i - x_0$ reduces equation 4.67 to

$$\Delta\,(\breve{x}_0)_{(k-1)} = \frac{1}{(k-1)!}\,(e_2-e_1)\wedge\cdots\wedge(e_k-e_1) \tag{4.68}$$

but equation 4.68 can be expanded into equation 4.69. The critical point in doing the expansion is that in generating the sum on the r.h.s. of the first line of equation 4.69 all products containing x_1 (of course all terms in the sum contain x_1 exactly once since we are using the outer product) are put in normal order by bringing the x_1 factor to the front of the product thus requiring the factor of $(-1)^i$ in each term in the sum.

$$\begin{aligned}(e_2-e_1)\wedge\cdots\wedge(e_k-e_1) &= e_2\wedge\cdots\wedge e_k - \sum_{i=2}^{k}(-1)^i\,e_1\wedge e_2\wedge\cdots\wedge\breve{e}_i\wedge\cdots\wedge e_k \\ &= \sum_{i=1}^{k}(-1)^{i-1}\,e_1\wedge e_2\wedge\cdots\wedge\breve{e}_i\wedge\cdots\wedge e_k \\ &= \sum_{i=1}^{k}e^iE_k \end{aligned} \tag{4.69}$$

or

$$\Delta\,(\breve{x}_0)_{(k-1)} = \frac{1}{(k-1)!}\sum_{i=1}^{k}e^iE_k \tag{4.70}$$

from equation 4.69 we have

$$\begin{aligned}
\mathsf{C} &= \frac{1}{(k-1)!}\sum_{i=1}^{k}\left(b\cdot\left(x_i-x_0\right)\right)e^iE_k \\
&= \frac{1}{(k-1)!}\sum_{i=1}^{k}\left(b\cdot e_i\right)e^iE_k \\
&= \frac{1}{(k-1)!}\sum_{i=1}^{k}b_ie^iE_k \\
&= \frac{1}{(k-1)!}bE_k \\
&= \frac{1}{(k-1)!}\left(b\cdot E_k+b\wedge E_k\right) \\
&= \frac{1}{(k-1)!}b\cdot E_k \qquad (4.71)
\end{aligned}$$

and equation 4.62 is proved which means substituting equation 4.62 into equation 4.61 proves equation 4.54

4.4.1 The Fundamental Theorem At Last!

The simplicial coordinates, λ^i, can be expressed in terms of the position vector, x, and the frame vectors of the simplex, $e_i = x_i - x_0$. Let the vectors e^j be the reciprocal frame to e_i ($e_i\cdot e^j = \delta_i^j$). Then

$$\lambda^i = e^i\cdot\left(x-x_0\right) \qquad (4.72)$$

and let $f\left(x\right)$ be an affine multivector function of x (equation 4.26) which interpolates, F, a differentiable multivector function of x on the simplex. Then

$$\begin{aligned}
\oint_{\partial(x)_{(k)}} f\left(x\right)\,dS &= \sum_{i=1}^{k}\left(F_i-F_0\right)\oint_{\partial(x)_{(k)}} e^i\cdot\left(x-x_0\right)dS \\
&= \sum_{i=1}^{k}\left(F_i-F_0\right)e^i\cdot\left(\Delta X\right) \qquad (4.73)
\end{aligned}$$

But

$$\frac{\partial f\left(x\right)}{\partial\lambda^i} = F_i - F_0 \qquad (4.74)$$

so that the surface integral of equation 4.73 can be rewritten

$$\begin{aligned}\oint_{\partial(x)_{(k)}} f(x)\ dS &= \sum_{i=1}^{k}(F_i - F_0)\, e^i\cdot(\Delta X)\\ &= \sum_{i=1}^{k}\frac{\partial f}{\partial\lambda^i}e^i\cdot(\Delta X) = \dot{f}\dot{\nabla}\cdot(\Delta X)\end{aligned}\tag{4.75}$$

If we now sum equation 4.75 over a chain of simplices realizing that the interpolated function $f(x)$ takes on the same value over the common boundary of two adjacent simplices, since $f(x)$ is only defined by the values at the common vertices. In forming a sum over a chain, all of the internal faces cancel and only the surface integral over the boundary remains. Thus

$$\oint f(x)\ dS = \sum_{a}\dot{f}\dot{\nabla}\cdot(\Delta X^a)\tag{4.76}$$

with the sum running over all simplices in the chain. Taking the limit as more points are added and each simplex is shrunk in size we obtain the first realization of the fundamental theorem of geometric calculus

$$\oint_{\partial V} F\ dS = \int_V \dot{F}\dot{\nabla}\ dX\tag{4.77}$$

We can write ∇dX instead of $\nabla\cdot dX$ since the vector ∇ is totally within the vector space defined by dX so that $\nabla\wedge dX = 0$. The method of proof used can also be applied to the form

$$\oint_{\partial V} dS\ G = \int_V \dot{\nabla}\ dX\ \dot{G}\tag{4.78}$$

A more general statement of the theorem is as follows:

Let $\mathsf{L}(A_{k-1}) = \mathsf{L}(A_{k-1};x)$ be a linear functional of a multivector A_{k-1} of grade $k-1$ and a general function of position x which returns a general multivector. The linear interpolation (approximation) of L over a simplex is defined by:

$$L(A) \equiv \mathsf{L}(A;x_0) + \sum_{i=1}^{k}\lambda^i\left(\mathsf{L}(A;x_i) - \mathsf{L}(A;x_0)\right)\tag{4.79}$$

Then the integral over a simplex is (Note that since integration is a linear operation, a summation, the integral can be placed inside $L(A)$ since $L(A)$ is linear in A)

$$
\begin{aligned}
\oint_{\partial(x)_{(k)}} L(dS) &= \mathsf{L}\left(\oint dS; x_0\right) + \sum_{i=1}^{k} \mathsf{L}\left(\oint \lambda^i dS; x_i\right) - \sum_{i=1}^{k} \mathsf{L}\left(\oint \lambda^i dS; x_0\right) \\
&= \sum_{i=1}^{k} \left(\mathsf{L}\left(e^i \Delta X; x_i\right) - \mathsf{L}\left(e^i \Delta X; x_0\right)\right) \\
&= \dot{L}\left(\dot{\nabla}\Delta X\right)
\end{aligned}
\tag{4.80}
$$

Taking the limit of a sum of simplicies gives

$$
\oint_{\partial V} \mathsf{L}(dS) = \int_V \dot{\mathsf{L}}\left(\dot{\nabla} dX\right) \tag{4.81}
$$

4.5 Examples of the Fundamental Theorem

4.5.1 Divergence and Green's Theorems

As a specific example consider $\mathsf{L}(A) = \langle JAI^{-1}\rangle$ where J is a vector, I is the unit pseudoscalar for a n-dimensional vector space, and A is a multivector of grade $n-1$. Then equation 4.81 gives

$$
\int_V \left\langle \dot{J}\dot{\nabla} dX I^{-1}\right\rangle = \int_V \nabla \cdot J\,|dX| = \oint_{\partial V} \left\langle J dS I^{-1}\right\rangle \tag{4.82}
$$

we have $dX = I\,|dX|$ where $|dX|$ is the scalar measure of the volume. The normal to the surface, n, is defined by

$$
n\,|dS| = dSI^{-1} \tag{4.83}
$$

where $|dS|$ is the scalar valued measure over the surface. With this definition we get

$$
\int_V \nabla \cdot J\,|dX| = \oint_{\partial V} n \cdot J\,|dS| \tag{4.84}
$$

Now using the form of the Fundamental Theorem of Geometric Calculus in equation 4.78 and let G be the vector $\boldsymbol{J}$ in two-dimensional Euclidean space and noting that since dA is a pseudoscalar (for 2-D ds is the boundary measure and dA is the volume measure) it anticommutes with vectors in two dimensions we get -

$$
\oint_{\partial A} ds\boldsymbol{J} = \int_A \dot{\boldsymbol{\nabla}} dA \dot{\boldsymbol{J}} = -\int_A \boldsymbol{\nabla}\boldsymbol{J} dA \tag{4.85}
$$

In 2-D Cartesian coordinates $dA = Idxdy$ and $ds = nI\left|ds\right|$ so that

$$\oint_{\partial A} ds\boldsymbol{J} = -\int_A \boldsymbol{\nabla J}Idxdy. \tag{4.86}$$

or

$$\begin{aligned} \oint_{\partial A} nI\boldsymbol{J}\left|ds\right| &= -\int_A \boldsymbol{\nabla J}Idxdy \\ -\oint_{\partial A} n\boldsymbol{J}I\left|ds\right| &= -\int_A \boldsymbol{\nabla J}Idxdy \qquad (4.87) \\ \oint_{\partial A} n\boldsymbol{J}\left|ds\right| &= \int_A \boldsymbol{\nabla J}dxdy \qquad (4.88) \end{aligned}$$

Letting $\boldsymbol{J} = P\boldsymbol{e}_x + Q\boldsymbol{e}_y$ and $n = n^x\boldsymbol{e}_x + n^y\boldsymbol{e}_y$ we get -

$$n\boldsymbol{J} = n\cdot\boldsymbol{J} + \left(n^xQ - n^yP\right)\boldsymbol{e}_x\boldsymbol{e}_y \tag{4.89}$$

$$\nabla\boldsymbol{J} = \nabla\cdot\boldsymbol{J} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{e}_x\boldsymbol{e}_y \tag{4.90}$$

so that

$$\oint_{\partial A} n\cdot\boldsymbol{J}\left|ds\right| = \int_A \nabla\cdot\boldsymbol{J}dxdy \tag{4.91}$$

$$\oint_{\partial A} \left(n^xQ - n^yP\right)\left|ds\right|\boldsymbol{e}_x\boldsymbol{e}_y = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy\boldsymbol{e}_x\boldsymbol{e}_y \tag{4.92}$$

but $dy = n^x\left|ds\right|$ and $dx = -n^y\left|ds\right|$ so that

$$\oint_{\partial A} Pdx + Qdy = \int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy \tag{4.93}$$

which is Green's theorem in the plane.

4.5.2 Cauchy's Integral Formula In Two Dimensions (Complex Plane)

Consider a two dimensional euclidean space with vectors $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$. Then the complex number z corresponding to $\boldsymbol{r}$ is

$$z = \boldsymbol{e}_x\boldsymbol{r} = x + y\boldsymbol{e}_x\boldsymbol{e}_y = x + yI \tag{4.94}$$

$$z^\dagger = \boldsymbol{r}\boldsymbol{e}_x = x - y\boldsymbol{e}_x\boldsymbol{e}_y = x - yI \tag{4.95}$$

$$zz^\dagger = z^\dagger z = x^2 + y^2 = \boldsymbol{r}^2 \tag{4.96}$$

Thus even grade multivectors correspond to complex numbers since $I^2 = -1$ and the reverse of z, $z^\dagger$ corresponds to the conjugate of z. Even grade multivectors commute with dS, since dS is proportional to I.

Let $\psi(\boldsymbol{r})$ be an even multivector function of $\boldsymbol{r}$, then

$$\int \boldsymbol{\nabla}\psi dS = \oint d\boldsymbol{s}\psi = \oint \frac{\partial \boldsymbol{r}}{\partial \lambda}\psi d\lambda \tag{4.97}$$

but the complex number z is given by $z = \boldsymbol{e}_x\boldsymbol{r}$ and

$$\boldsymbol{e}_x \oint d\boldsymbol{s}\psi = \oint \psi dz = \int \boldsymbol{e}_x\boldsymbol{\nabla}\psi dS. \tag{4.98}$$

Thus if a function ψ is analytic, $\boldsymbol{\nabla}\psi = 0$ and $\oint \psi dz = 0$. Now note that (this will be proved for the N-dimensional case in the next example)

$$\boldsymbol{\nabla}\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2} = 2\pi\delta(\boldsymbol{r}-\boldsymbol{a}) \tag{4.99}$$

where $a = \boldsymbol{e}_x\boldsymbol{a}$. Now let

$$\psi = \frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\boldsymbol{e}_x f(\boldsymbol{e}_x\boldsymbol{r}) \tag{4.100}$$

so that

$$\begin{aligned}
\boldsymbol{e}_x \oint d\boldsymbol{s}\psi &= \boldsymbol{e}_x \oint d\boldsymbol{s}\left(\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\boldsymbol{e}_x f(\boldsymbol{e}_x\boldsymbol{r})\right) \\
&= \oint dz\left(\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\boldsymbol{e}_x f(\boldsymbol{e}_x\boldsymbol{r})\right) \\
&= \oint dz\left(\frac{(z-a)^\dagger}{(z-a)(z-a)^\dagger} f(z)\right) \\
&= \oint \frac{f(z)}{z-a}dz
\end{aligned} \tag{4.101}$$

and

$$\begin{aligned}
\oint \frac{f(z)}{z-a}dz &= \boldsymbol{e}_x \int \boldsymbol{\nabla}\left(\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\boldsymbol{e}_x f(\boldsymbol{e}_x\boldsymbol{r})\right) dS \\
&= \boldsymbol{e}_x \int \left(2\pi\delta(\boldsymbol{r}-\boldsymbol{a})\,\boldsymbol{e}_x f(\boldsymbol{e}_x\boldsymbol{r}) + \boldsymbol{\nabla} f(\boldsymbol{e}_x\boldsymbol{r})\,\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\boldsymbol{e}_x\right) I\,|dS| \\
&= 2\pi I f(a) + \int \boldsymbol{e}_x\boldsymbol{\nabla} f(\boldsymbol{e}_x\boldsymbol{r})\,\frac{z^\dagger - a^\dagger}{|z-a|^2} I\,|dS| \\
&= 2\pi I f(a) + \int \boldsymbol{e}_x\boldsymbol{\nabla} f(\boldsymbol{e}_x\boldsymbol{r})\,\frac{1}{z-a} I\,|dS| \\
&= 2\pi I f(a) + \int \left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f(z)\,\frac{1}{z-a} I\,|dS| \qquad (4.102)
\end{aligned}$$

If $\boldsymbol{\nabla} f(z) = 0$ we have -

$$\oint \frac{f(z)}{z-a}dz = 2\pi I f(a) \qquad (4.103)$$

which is the Cauchy integral formula. If $\boldsymbol{\nabla} f(z)$ is not zero we can write the more general relation -

$$2\pi I f(a) = \oint \frac{f}{z-a}dz - 2\int \frac{\partial f}{\partial z^\dagger}\frac{1}{z-a} I\,|dS| \qquad (4.104)$$

since

$$\frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) \qquad (4.105)$$

and

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) \qquad (4.106)$$

4.5.3 Green's Functions in N-dimensional Euclidean Spaces

Let ψ be an even multivector function or let N be even so that ψ commutes with I. The analog of an analytic function in N-dimensions is $\nabla\psi = 0$.

The Green's function of the ∇ operator is ($S_N = 2\pi^{N/2}/\Gamma(N/2)$ is the hyperarea of the unit radius sphere in N dimensions)

$$G(x;y) = \lim_{\epsilon\to 0} \frac{x-y}{S_N\left(|x-y|^N + \epsilon\right)} \qquad (4.107)$$

So that

$$\lim_{\epsilon\to 0}\nabla_x G\left(x;y\right) = \delta\left(x-y\right). \tag{4.108}$$

To prove equation 4.108 we need to use

$$\nabla_x\left(x-y\right) = N \quad\text{and}\quad \nabla_x\left|x-y\right|^M = M\left(x-y\right)\left|x-y\right|^{M-2}$$

So that

$$\begin{aligned}
\nabla_x G &= \frac{1}{S_N}\left\{\nabla_x\left(\left|x-y\right|^N+\epsilon\right)^{-1}\left(x-y\right)+\left(\left|x-y\right|^N+\epsilon\right)^{-1}\nabla_x\left(x-y\right)\right\}\\
&= \frac{N}{S_N}\left\{\frac{-\left|x-y\right|^N}{\left(\left|x-y\right|^N+\epsilon\right)^2}+\frac{1}{\left(\left|x-y\right|^N+\epsilon\right)}\right\}\\
&= \frac{N}{S_N}\frac{\epsilon}{\left(\left|x-y\right|^N+\epsilon\right)^2} = \nabla_x\cdot G = \dot{G}\cdot\dot{\nabla}_x = \dot{G}\dot{\nabla}_x
\end{aligned} \tag{4.109}$$

so what must be proved is that

$$\lim_{\epsilon\to 0}\frac{N}{S_N}\frac{\epsilon}{\left(\left|x-y\right|^N+\epsilon\right)^2} = \delta\left(x-y\right) \tag{4.110}$$

First define the volume B_τ $(\tau > 0)$ by

$$x\in B_\tau \iff \left|x\right|\le\tau \tag{4.111}$$

and let $y = 0$ and calculate (Note that we use $|dV|$ since in our notation $dV = I\,|dV|$ and the oriented dV is not needed in this proof. Also $r = |x|$.)

$$\begin{aligned}
\int_{B_\infty}\frac{N}{S_N}\frac{\epsilon}{\left(\left|x\right|^N+\epsilon\right)^2}\left|dV\right| &= \int_0^\infty\frac{N\epsilon r^{N-1}}{\left(r^N+\epsilon\right)^2}dr\\
&= \int_0^\infty\frac{\epsilon}{\left(r^N+\epsilon\right)^2}d\left(r^N\right)\\
&= -\left[\frac{\epsilon}{r^N+\epsilon}\right]_0^\infty\\
&= 1
\end{aligned} \tag{4.112}$$

Thus the first requirement of a delta function is fulfilled. Now let $\phi(x)$ be a scalar test function on the N-dimensional space and S a point set in the space and define the functions

$$\max\left(\phi, S\right) = \left\{\max\left(\phi(x)\right) \forall x \in S\right\} \quad \text{and} \quad \min\left(\phi, S\right) = \left\{\min\left(\phi(x)\right) \forall x \in S\right\}$$

and calculate the integral

$$\begin{aligned} \frac{N}{S_N}\int_{B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| &= \int_0^\tau \frac{\epsilon}{\left(r^N + \epsilon\right)^2} d\left(r^N\right) \\ &= -\left[\frac{\epsilon}{r^N + \epsilon}\right]_0^\tau \\ &= 1 - \frac{\epsilon}{\tau^N + \epsilon} \end{aligned} \tag{4.113}$$

and note that

$$\frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| = \frac{\epsilon}{\tau^N + \epsilon} \tag{4.114}$$

Thus $\forall \tau > 0$ we have

$$\lim_{\epsilon \to 0} \frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| = 0 \tag{4.115}$$

and

$$\lim_{\epsilon \to 0} \frac{N}{S_N}\int_{B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| = 1 \tag{4.116}$$

Thus

$$\begin{aligned} &\min\left(\phi, B_\infty - B_\tau\right) \frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| \leq \\ &\frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon\phi(x)}{\left(|x|^N + \epsilon\right)^2}\,|dV| \\ &\leq \max\left(\phi, B_\infty - B_\tau\right) \frac{N}{S_N}\int_{B_\infty - B_\tau} \frac{\epsilon}{\left(|x|^N + \epsilon\right)^2}\,|dV| \end{aligned} \tag{4.117}$$

and

$$\begin{aligned}\min\left(\phi,B_\tau\right)\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon}{\left(|x|^N+\epsilon\right)^2}\left|dV\right| \leq \\ \frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\left|dV\right| \\ \leq \max\left(\phi,B_\tau\right)\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon}{\left(|x|^N+\epsilon\right)^2}\left|dV\right|\end{aligned} \tag{4.118}$$

Thus

$$\lim_{\epsilon\rightarrow 0}\frac{N}{S_N}\int_{B_\infty-B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\left|dV\right| = 0 \tag{4.119}$$

and

$$\min\left(\phi,B_\tau\right)\leq\lim_{\epsilon\rightarrow 0}\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\left|dV\right|\leq\max\left(\phi,B_\tau\right) \tag{4.120}$$

Finally

$$\lim_{\tau\rightarrow 0}\lim_{\epsilon\rightarrow 0}\frac{N}{S_N}\int_{B_\tau}\frac{\epsilon\phi(x)}{\left(|x|^N+\epsilon\right)^2}\left|dV\right| = \phi(0) \tag{4.121}$$

and we have proved equation 4.110 since

$$\lim_{\tau\rightarrow 0}\left(\max(\phi,B_\tau)-\min(\phi,B_\tau)\right) = 0 \tag{4.122}$$

Now in the fundamental theorem of geometric calculus let $\mathsf{L}(A) = GA\psi$ so that (remember that ψ commutes with dV and $\nabla_x\psi = 0$)

$$\begin{aligned}\dot{\mathsf{L}}\left(\dot{\nabla}_x dV\right) &= \dot{G}\dot{\nabla}_x dV\psi + G\dot{\nabla}_x dV\dot{\psi} \\ &= \left(\dot{G}\dot{\nabla}_x\psi + G\nabla_x\psi\right)I\left|dV\right| \\ &= \dot{G}\dot{\nabla}_x\psi I\left|dV\right| \\ &= \nabla_x G\psi I\left|dV\right|\end{aligned} \tag{4.123}$$

and

$$\begin{aligned}\oint_{\partial V} G dS \psi &= \int_V \delta(x-y)\,\psi(x)\,I\,|dV| \\ &= \psi(y)\,I \\ &= I\psi(y) \end{aligned} \tag{4.124}$$

or

$$\psi(y) = \frac{I^{-1}}{S_N}\oint_{\partial V} \frac{x-y}{|x-y|^N} dS \psi(x) \tag{4.125}$$

because ψ is a monogenic function ($\nabla\psi = 0$).

Chapter 5

Geometric Calculus on Manifolds

5.1 Definition of a Vector Manifold

The definition of a manifold that we will use is -

A vector manifold (generalized surface) is a set of points labeled by vectors lying in a geometric algebra of arbitra dimension and signature. If we consider a path in the surface $x(\lambda)$*, the tangent vector is defined by*

$$x' \equiv \left.\frac{\partial x(\lambda)}{\partial \lambda}\right|_{\lambda_0} = \lim_{\epsilon \to 0} \frac{x(\lambda_0+\epsilon) - x(\lambda_0)}{\epsilon} \qquad (5$$

and the path length

$$s \equiv \int_{\lambda_1}^{\lambda_2} \sqrt{|x' \cdot x'|}\, d\lambda \qquad (5$$

5.1.1 The Pseudoscalar of the Manifold

Now introduce a set of paths in the surface all passing through the same point x. These paths define a set of tangent vectors $\{e_1, \ldots, e_n\}$. We assume these paths have been picked so that the vectors are independent and form a basis for the tangent space at point x. The outer product of the tangent vectors form the pseudoscalar, $I(x)$, for the tangent space

$$I(x) \equiv \frac{e_1 \wedge e_2 \wedge \cdots \wedge e_n}{|e_1 \wedge e_2 \wedge \cdots \wedge e_n|} \tag{5.3}$$

Thus

$$I^2 = \pm 1 \tag{5.4}$$

We require that for any point on the manifold the denominator of equation 5.3 is nonzero. We also assume that for all points on the manifold that $I(x)$ is continuous, differentiable, singled valued, and has the same grade everywhere.

5.1.2 The Projection Operator

Define the projection operator $\mathcal{P}(A)$ operating on any multivector A in the embedding multivector space as

$$\mathcal{P}(A) \equiv (A \cdot I(x))\, I^{-1}(x) \tag{5.5}$$

We can show that $\mathcal{P}(A)$ extracts those components of A that lie in the geometric algebra defined by $I(x)$. Since $\mathcal{P}(A)$ is linear in A if we show that if $\mathcal{P}(A_r)$ projects correctly for an r-grade multivector it will do so for a general mulitvector. If n is the dimension of the tangent space and A_r is a pure grade multivector we can write equation 5.5 as

$$\mathcal{P}(A_r) = \langle A_r I(x) \rangle_{|r-n|}\, I^{-1}(x) \tag{5.6}$$

Now consider the blades that make up the components of A_r. They will either consist only of blades formed from the tangent vectors e_i or they will contain at least one basis vector that is not a tangent vector. In the first case

$$\langle A_r I(x) \rangle_{|r-n|} = A_r I(x) \tag{5.7}$$

and

$$\mathcal{P}(A_r) = A_r \tag{5.8}$$

In the second case there is no component of $A_r I(x)$ of grade $|r-n|$ and

$$\mathcal{P}(A_r) = 0 \tag{5.9}$$

This is easily seen if one constructs at point x an orthogonal basis ($o_i \cdot o_j = \delta_{ij} o_i^2$) for the tangent space $\{o_1, \ldots, o_n\}$ and an orthogonal basis for the remainder of the embedding space $\{o_{n+1}, \ldots, o_m\}$. Then any component blade of A_r is of the form

$$A_r^{i_1,i_2,\ldots,i_r} o_{i_1} o_{i_2} \ldots o_{i_r} \tag{5.10}$$

where $i_1 < i_2 < \cdots < i_r$. If $i_j \le n \,\forall\, 1 \le j \le r$ then

$$A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} \cdot I = A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} I \tag{5.11}$$

and

$$\left(A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} \cdot I\right) I^{-1} = A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} \tag{5.12}$$

and

$$\mathcal{P}\left(A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r}\right) = A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} \tag{5.13}$$

If any $i_j > m$ then $A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r} I$ contains no grade $|r-n|$ and

$$\mathcal{P}\left(A_r^{i_1,i_2,\dots,i_r} o_{i_1} o_{i_2} \dots o_{i_r}\right) = 0 \tag{5.14}$$

5.1.3 The Exclusion Operator

For an arbitrary multivector A the exclusion operator $\mathcal{P}_\perp(A)$ is defined by

$$\mathcal{P}_\perp(A) \equiv A - \mathcal{P}(A) \tag{5.15}$$

5.1.4 The Intrinsic Derivative

Given a set of tangent vectors $\{e_i\}$ spanning the tangent space and the geometric derivative, ∇, for the embedding space, the derivative intrinsic to the manifold is defined everywhere by

$$\partial \equiv e^i e_i \cdot \nabla = \mathcal{P}(\nabla) \tag{5.16}$$

Also note that

$$\mathcal{P}(\partial) = \partial \tag{5.17}$$

When we write $\mathcal{P}(\nabla)$ or $\mathcal{P}(\partial)$ the ∇ or ∂ is not differentiating the $I(x)$ in the P operator anymore than ∇ is differentiating dX in the fundamental theorem of Geometric Calculus.

We also note that if the vector a is in the tangent space that

$$a \cdot \partial = a \cdot \nabla \tag{5.18}$$

and that $a \cdot \partial$ is a scalar operator that gives the directional derivative in the a direction. Also since it is scalar it satisfies Leibniz's rules without using the dot notation **(remember the convention**

that if parenthesis are not present the operator precedence is dot product then wedge product then geometric product).

$$a\cdot\partial\left(AB\right)=\left(a\cdot\partial A\right)B+A\left(a\cdot\partial B\right) \tag{5.19}$$

An alternative definition for the intrinsic derivative[1] is to let $\gamma\left(s\right)$ be a curve on the manifold. Then $\dfrac{d\gamma}{ds}$ is a tangent vector to the mainfold and we can define

$$\left.\frac{d\gamma}{ds}\right|_{s=s_0}\cdot\partial\left.A\right|_{s=s_0}\equiv\left.\frac{dA\left(\gamma\left(s\right)\right)}{ds}\right|_{s=s_0}. \tag{5.20}$$

We can show that equation 5.20 is equivalent to equation 5.16 with the following construction. Let $\left(x^1,\dots,x^n\right)$ be a local coordinate system for the vector manifold $x=x\left(x^1,\dots,x^n\right)$. Then a basis for the tangent space is $e_i=\dfrac{\partial x}{\partial x^i}$ and the intrinsic derivative is $\partial=e^i\dfrac{\partial}{\partial x^i}$. Now write

$$A\left(\gamma\left(s\right)\right)=A\left(x^1\left(s\right),\dots,x^n\left(s\right)\right) \tag{5.21}$$

so that

$$\frac{dA\left(\gamma\left(s\right)\right)}{ds}=\frac{\partial A}{\partial x^i}\frac{dx^i}{ds} \tag{5.22}$$

but

$$\frac{d\gamma}{ds}=\frac{\partial x}{\partial x^i}\frac{dx^i}{ds}=e_i\frac{dx^i}{ds} \tag{5.23}$$

so that

$$\frac{d\gamma}{ds}\cdot\partial A=e_i\frac{dx^i}{ds}\cdot e^j\frac{\partial A}{\partial x^j}=\frac{dx^i}{ds}\frac{\partial A}{\partial x^i}=\frac{dA\left(\gamma\left(s\right)\right)}{ds} \tag{5.24}$$

and the two definitions are equivalent.

5.1.5 The Covariant Derivative

The ∂ operator is entirely within the tangent space and if the general multivector function $A\left(x\right)$ is also entirely within the tangent space, it is still possible (even likely) that ∂A is not entirely within the tangent space. We need a covariant derivative D that will result in a multivector entirely within the tangent space. This can be done by defining

$$a\cdot DA\left(x\right)\equiv\mathcal{P}\left(a\cdot\partial A\left(x\right)\right) \tag{5.25}$$

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[1] Private conversation with Dr. Alan MacDonald.

so that

$$a\cdot\partial A=\mathcal{P}\left(a\cdot\partial A\right)+\mathcal{P}_\perp\left(a\cdot\partial A\right)=a\cdot DA+\mathcal{P}_\perp\left(a\cdot\partial A\right)\tag{5.26}$$

Again since $a\cdot D$ is a scalar operator we have

$$a\cdot D\left(AB\right)=\mathcal{P}\left(a\cdot\partial\left(AB\right)\right)=\left(a\cdot DA\right)B+A\left(a\cdot DB\right)\tag{5.27}$$

A component expansion of D is given in the usual way by (do not forget the summation convention)

$$D=e^ie_i\cdot D\tag{5.28}$$

and

$$DA_r=e^i\left(e_i\cdot DA_r\right)=\mathcal{P}\left(\partial A_r\right)\tag{5.29}$$

and

$$D\cdot A_r\equiv\left\langle DA_r\right\rangle_{r-1}\tag{5.30}$$

$$D\wedge A_r\equiv\left\langle DA_r\right\rangle_{r+1}\tag{5.31}$$

if $\alpha\left(x\right)$ is a scalar function on the manifold then

$$\partial\alpha\left(x\right)=D\alpha\left(x\right)\tag{5.32}$$

because in equation 5.32 no basis vectors are differentiated. To relate ∂ and D if the function operated on is not a scalar first construct a normalized basis $\{e_i\}$ for the tangent space at point x. Then

$$I=e_1\wedge e_2\wedge\ldots\wedge e_n\text{ and }I^2=\pm1\tag{5.33}$$

and (since $a\cdot\partial$ and $a\cdot D$ are scalar operators we can move them across the wedge products without any problems)

$$\left(a\cdot\partial I\right)I^{-1}=\left(\sum_{i=1}^{n}e_1\wedge\ldots\wedge\left(a\cdot De_i+\mathcal{P}_\perp\left(a\cdot\partial e_i\right)\right)\wedge\ldots\wedge e_n\right)I^{-1}\tag{5.34}$$

$$=\left(a\cdot DI\right)I^{-1}+\sum_{i=1}^{n}\left(-1\right)^{i-1}\mathcal{P}_\perp\left(a\cdot\partial e_i\right)\wedge e_1\wedge\ldots\wedge\breve{e}_i\wedge\ldots\wedge e_nI^{-1}\tag{5.35}$$

$$=\left(a\cdot DI\right)I^{-1}+\mathcal{P}_\perp\left(a\cdot\partial e_i\right)\wedge e^i\tag{5.36}$$

We go from equation 5.35 to equation 5.36 by using equation 1.61 on page 33 in the section on reciprocal frames.

Since $(a\cdot D)\,I$ is a grade n multivector in the tangent space it must be proportional to I and thus commute with I so that $((a\cdot\partial)\,I)\,I = I\,((a\cdot\partial)\,I)$. Also $I^{-1} = \pm I$ so that we have

$$\begin{aligned}
(a\cdot DI)\,I^{-1} &= \pm\,(a\cdot DI)\,I \\
&= \pm\frac{1}{2}\,((a\cdot DI)\,I + I\,(a\cdot DI)) \\
&= \pm\frac{1}{2}\left(a\cdot D\left(I^2\right)\right) \\
&= 0
\end{aligned} \tag{5.37}$$

Thus

$$(a\cdot\partial I) = \mathcal{P}_\perp\,(a\cdot\partial e_i)\wedge e^i I \equiv -S\,(a)\,I \tag{5.38}$$

Where $S\,(a)$ is the shape tensor associated with the manifold. Since $S\,(a)$ is a bivector we can write $(A\times B = (AB - BA)\,/2)$

$$a\cdot\partial I = I\times S\,(a) \tag{5.39}$$

since

$$S\,(a)\cdot I = S\,(a)\wedge I = 0 \tag{5.40}$$

and by equation 1.96 page 58. Given that $a\,(x)$ and $b\,(x)$ are vector fields on the manifold (both are in the tangent space at point x), form the expressions in equation 5.41 (remember that for any three vectors u, v, and w we have $u\cdot(v\wedge w) = (u\cdot v)\,w - (u\cdot w)\,v$)

$$\begin{aligned}
b\cdot S\,(a) &= b\cdot\left(e^i\wedge\mathcal{P}_\perp\,(a\cdot\partial e_i)\right) \\
&= \left(b\cdot e^i\right)\mathcal{P}_\perp\,(a\cdot\partial e_i) - (b\cdot\mathcal{P}_\perp\,(a\cdot\partial e_i))\,e^i \\
&= \left(\left(b^j e_j\right)\cdot e^i\right)\mathcal{P}_\perp\,(a\cdot\partial e_i) \\
&= b^j\delta^i_j\mathcal{P}_\perp\,(a\cdot\partial e_i) \\
&= \mathcal{P}_\perp\left(a\cdot\dot{\partial} b^i\dot{e}_i\right)
\end{aligned} \tag{5.41}$$

but

$$\begin{aligned}
\mathcal{P}_\perp\,(a\cdot\partial b) &= \mathcal{P}_\perp\left(a\cdot\partial\left(b^i e_i\right)\right) \\
&= \mathcal{P}_\perp\left(a\cdot\dot{\partial}\dot{b}^i e_i + a\cdot\dot{\partial} b^i\dot{e}_i\right) \\
&= \mathcal{P}_\perp\left(a\cdot\dot{\partial} b^i\dot{e}_i\right)
\end{aligned} \tag{5.42}$$

and

$$\boxed{b\cdot S\,(a) = \mathcal{P}_\perp\,(a\cdot\partial b)} \tag{5.43}$$

Thus

$$a\cdot\partial b=\mathcal{P}\left(a\cdot\partial b\right)+\mathcal{P}_{\perp}\left(a\cdot\partial b\right)=a\cdot Db+b\cdot S\left(a\right) \tag{5.44}$$

and (using the fact that the dot product of a vector and bivector are antisymmetric)

$$a\cdot Db=a\cdot\partial b+S\left(a\right)\cdot b \tag{5.45}$$

Now consider the expression

$$\begin{aligned}
a\cdot D\left(b_1\dots b_r\right) &= \sum_{i=1}^{r}b_1\dots\left(a\cdot\partial b_i+S\left(a\right)\cdot b_i\right)\dots b_r\\
&= \sum_{i=1}^{r}b_1\dots\left(a\cdot\partial b_i\right)\dots b_r+\\
&\qquad\qquad \sum_{i=1}^{r}b_1\dots\left(S\left(a\right)\cdot b_i\right)\dots b_r\\
&= a\cdot\partial\left(b_1\dots b_r\right)+\\
&\qquad\qquad \frac{1}{2}\sum_{i=1}^{r}b_1\dots\left(S\left(a\right)b_i-b_iS\left(a\right)\right)\dots b_r\\
&= a\cdot\partial\left(b_1\dots b_r\right)+\frac{1}{2}\left(S\left(a\right)\left(b_1\dots b_r\right)-\left(b_1\dots b_r\right)S\left(a\right)\right)+\\
&\qquad\qquad \frac{1}{2}\left(\sum_{i=2}^{r-1}b_1\dots S\left(a\right)b_i\dots b_r-\sum_{i=2}^{r-1}b_1\dots b_iS\left(a\right)\dots b_r\right)+\\
&\qquad\qquad \frac{1}{2}\left(b_1\dots b_{r-1}S\left(a\right)b_r-b_1S\left(a\right)b_2\dots b_r\right)\\
&= a\cdot\partial\left(b_1\dots b_r\right)+\frac{1}{2}\left(S\left(a\right)\left(b_1\dots b_r\right)-\left(b_1\dots b_r\right)S\left(a\right)\right)+\\
&\qquad\qquad \frac{1}{2}\left(\sum_{i=3}^{r-1}b_1\dots S\left(a\right)b_i\dots b_r-\sum_{i=2}^{r-2}b_1\dots b_iS\left(a\right)\dots b_r\right)+ &\quad(5.46)\\
&= a\cdot\partial\left(b_1\dots b_r\right)+S\left(a\right)\times\left(b_1\dots b_r\right) &\quad(5.47)
\end{aligned}$$

To get from equation 5.46 to equation 5.47 note that in the sums in parenthesis in equation 5.46 the i^{th} term in the first sum cancels the $i^{th}+1$ term in the second sum.

Since any multivector is a linear superposition of terms containing $b_1\dots b_r$ with $1\leq r\leq n$ and a scalar we have

$$\boxed{a\cdot DA=a\cdot\partial A+S\left(a\right)\times A} \tag{5.48}$$

Where $a\left(x\right)$ and $b\left(x\right)$ are vector fields on the manifold write

$$a\cdot\partial b = a\cdot\partial\mathcal{P}\left(b\right) = a\cdot\dot{\partial}\dot{\mathcal{P}}\left(b\right)+\mathcal{P}\left(a\cdot\partial b\right) = a\cdot\dot{\partial}\dot{\mathcal{P}}\left(b\right)+a\cdot Db \tag{5.49}$$

Now substitute equation 5.48 into equation 5.49 to get

$$a\cdot\dot{\partial}\dot{\mathcal{P}}\left(b\right) = b\cdot S\left(a\right) \tag{5.50}$$

5.2 Coordinates and Derivatives

In a region of the manifold we introduce local coordinates x^i and define the frame vectors as

$$e_i = \frac{\partial x}{\partial x^i} \tag{5.51}$$

From the definition of ∂ is follows that $e^i = \partial x^i$. The $\{e_i\}$ are referred to as tangent vectors and the reciprocal frame $\{e^i\}$ as the cotangent vector (or 1-forms). The covariant derivative along a coordinate vector, $e_i\cdot D$, satisfies and defines both D_i and S_i.

$$\boxed{e_i\cdot DA = D_iA = e_i\partial A + S\left(e_i\right)\times A \equiv \partial_iA + S_i\times A} \tag{5.52}$$

The tangent frame vectors satisfy

$$\partial_ie_j - \partial_je_i = \left(\partial_i\partial_j - \partial_j\partial_i\right)x = 0 \tag{5.53}$$

Using the $\mathcal{P}$ operator on equation 5.53 gives

$$D_ie_j - D_je_i = 0 \tag{5.54}$$

while using $\mathcal{P}_\perp$ gives

$$e_i\cdot S_j = e_j\cdot S_i \tag{5.55}$$

For arbitrary vectors a and b in the tangent space equation 5.55 becomes

$$a\cdot S\left(b\right) = b\cdot S\left(a\right) \tag{5.56}$$

In terms of the coordinate vectors the shape tensor becomes

$$\boxed{S\left(a\right) = e^k\wedge\mathcal{P}_\perp\left(a\cdot\partial e_k\right)} \tag{5.57}$$

and

$$S_i = e^k \wedge \mathcal{P}_\perp (e_i \cdot \partial e_k) = e^k \wedge \mathcal{P}_\perp (e_k \cdot \partial e_i) \tag{5.58}$$

Then

$$\partial \wedge e_i = e^k \wedge \partial_k e_i = e^k \wedge (\mathcal{P}(\partial_k e_i) + \mathcal{P}_\perp (\partial_k e_i)) = D \wedge e_i + S_i \tag{5.59}$$

Letting $a = a^i e_i$ be a constant vector in the tangent space gives the general result

$$\boxed{\partial \wedge a = D \wedge a + S(a)} \tag{5.60}$$

Additionally

$$\begin{aligned}\partial \wedge a &= \partial \wedge (\mathcal{P}(a)) \\ &= \dot{\partial} \wedge \dot{\mathcal{P}}(a) + \mathcal{P}(\partial \wedge a) \\ &= D \wedge a + \dot{\partial} \wedge \dot{\mathcal{P}}(a)\end{aligned} \tag{5.61}$$

Thus

$$\dot{\partial} \wedge \dot{\mathcal{P}}(a) = S(a) \tag{5.62}$$

Note that if a and b are any two vectors in the embedding space then $\mathcal{P}(a \wedge b) = \mathcal{P}(a) \wedge \mathcal{P}(b)$ and if $\phi(x)$ is a scalar function on the manifold we have

$$\begin{aligned}\partial \wedge \partial\phi &= \partial \wedge \mathcal{P}(\nabla\phi) \\ &= \dot{\partial} \wedge \mathcal{P}\left(\dot{\nabla}\phi\right) + \dot{\partial} \wedge \dot{\mathcal{P}}(\nabla\phi) \\ &= \mathcal{P}\left(\dot{\nabla}\right) \wedge \mathcal{P}\left(\dot{\nabla}\phi\right) + \dot{\partial} \wedge \dot{\mathcal{P}}(\nabla\phi) \\ &= \mathcal{P}(\nabla \wedge \nabla\phi) + \dot{\partial} \wedge \dot{\mathcal{P}}(\nabla\phi)\end{aligned} \tag{5.63}$$

but $\nabla \wedge \nabla = 0$ so

$$\partial \wedge \partial\phi = S(\nabla\phi) \tag{5.64}$$

Since $S(a)$ for any vector a lies outside the manifold we have

$$D \wedge (D\phi) = 0 \tag{5.65}$$

Letting $\phi(x) = x^i(x)$, then since $x^i(x)$ is a scalar function

$$D \wedge \left(Dx^i\right) = D \wedge e^i = 0 \tag{5.66}$$

so that for a general vector $a = a_i(x)\, e^i$ we have

$$D \wedge a = D \wedge \left(a_j e^j\right) = e^i \wedge e^j (\partial_i a_j) = \frac{1}{2} e^i \wedge e^j (\partial_i a_j - \partial_j a_i) \tag{5.67}$$

Equation 5.67 is isomorphic to the definition of the *exterior derivative* of differential geometry.

5.3 Riemannian Geometry

We shall now relate the shape tensor to the metric tensor and Christoffel connection. The metric tensor is defined by

$$g_{ij} \equiv e_i \cdot e_j \tag{5.68}$$

and the Christoffel connection by

$$\Gamma^i_{jk} \equiv (D_j e_k) \cdot e^i \tag{5.69}$$

so that the components of the covariant derivative are given by

$$\begin{aligned}(a \cdot Db) \cdot e^i &= a^j \left(D_j \left(b^k e_k\right)\right) \cdot e^i \\ &= a^j \left(\partial_j b^i + \Gamma^i_{jk} b^k\right) \end{aligned} \tag{5.70}$$

The Γ^i_{jk} can be expressed in terms of the g_{ij} by considering the following relations. First, the Γ^i_{jk} are symmetric in the j and k indices.

$$\Gamma^i_{jk} - \Gamma^i_{kj} = (D_j e_k - D_k e_j) \cdot e^i = 0 \tag{5.71}$$

Second, the curl of the basis vectors is given by (equation 5.66)

$$D \wedge e_i = D \wedge \left(g_{ij} e^j\right) = (D g_{ij}) \wedge e^j \tag{5.72}$$

By equation 5.71 we can write

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2} e^i \cdot (D_j e_k + D_k e_j) \\ &= \frac{1}{2} e^i \cdot ((e_j \cdot D)\, e_k + (e_k \cdot D)\, e_j) \end{aligned} \tag{5.73}$$

Now apply equation B.6 (Appendix A) and equation 5.72 to each term in equation 5.73 to get

$$\begin{aligned}(e_j \cdot D)\, e_k + (e_k \cdot D)\, e_j &= e_j \cdot (D \wedge e_k) + e_k \cdot (D \wedge e_j) + (e_j \cdot \dot{e}_k)\, \dot{D} + (e_k \cdot \dot{e}_j)\, \dot{D} \\ &= e_j \cdot (D \wedge e_k) + e_k \cdot (D \wedge e_j) + D\,(g_{jk}) \\ &= e_j \cdot \left((D g_{kl}) \wedge e^l\right) + e_k \cdot \left((D g_{jl}) \wedge e^j\right) + D\,(g_{jk}) \end{aligned} \tag{5.74}$$

Now apply equation B.2 (Appendix A) to $e_j \cdot \left(D g_{kl} \wedge e^l\right)$ and $e_k \cdot (D g_{jl} \wedge e^j)$ giving in the first case

$$\begin{aligned}e_j \cdot \left(D g_{kl} \wedge e^l\right) &= (e_j \cdot (D g_{kl}))\, e^l - \left(e_j \cdot e^l\right) D g_{kl} \\ &= ((e_j \cdot D)\, g_{kl})\, e^l - \delta^l_j D g_{kl} \\ &= (D_j g_{kl})\, e^l - D g_{kj} \\ &= (\partial_j g_{kl})\, e^l - \partial g_{kj} \end{aligned} \tag{5.75}$$

so that equation 5.73 becomes

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}e^i \cdot \left(\left(\partial_j g_{kl}\right) e^l + \left(\partial_k g_{jl}\right) e^l - \partial g_{kj} \right) \\ &= \frac{1}{2}e^i \cdot \left(\left(\partial_j g_{kl}\right) e^l + \left(\partial_k g_{jl}\right) e^l - e^l \partial_l g_{kj} \right) \\ &= \frac{1}{2}g^{il} \left(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{kj}\right) \end{aligned} \tag{5.76}$$

which is the standard formula for the Γ^i_{jk}.

Now define the commutator bracket $[A, B]$ of the multivectors A and B by (note there is no $\frac{1}{2}$ factor)

$$[A, B] \equiv AB - BA \tag{5.77}$$

Now form equation B.2 (Appendix A) and use the Jacobi identity (equation 1.100) to reduce the double commutator products on the r.h.s. of the equation

$$\begin{aligned}[D_i, D_j] A &= \partial_i \left(\partial_j A + S_j \times A\right) + S_i \times \left(\partial_j A + S_j \times A\right) - \partial_j \left(\partial_i A + S_i \times A\right) - S_j \times \left(\partial_i A + S_i \times A\right) \\ &= \left(\partial_i S_j - \partial_j S_i\right) \times A + \left(S_i \times S_j\right) \times A \end{aligned} \tag{5.78}$$

However (see equations 5.38 and 5.39, page 153) so that $S_i = -\left(\partial_i I\right) I^{-1}$, and

$$\begin{aligned}\left(\partial_i S_j - \partial_j S_i\right) &= -\partial_i \left(\left(\partial_j I\right) I^{-1} \right) + \partial_j \left(\left(\partial_i I\right) I^{-1} \right) \\ &= \left(\partial_j I\right) \left(\partial_i I\right) I^{-2} - \left(\partial_i I\right) \left(\partial_j I\right) I^{-2} \\ &= \left(\partial_j I\right) I^{-1} \left(\partial_i I\right) I^{-1} - \left(\partial_i I\right) I^{-1} \left(\partial_j I\right) I^{-1} \\ &= S_j S_i - S_i S_j = -2 S_i \times S_j \end{aligned} \tag{5.79}$$

where we have used that I^{-1} and the partial derivatives of I commute to reduce the second line of equation 5.79

$$[D_i, D_j] A = -\left(S_i \times S_j\right) \times A \tag{5.80}$$

The commutator of the covariant derivatives defines the Riemann tensor

$$\mathcal{R}\left(a, b\right) \equiv \mathcal{P}\left(S\left(b\right) \times S\left(b\right)\right) \tag{5.81}$$

Since $\mathcal{R}\left(a, b\right)$ is a bilinear antisymmetric function of a and b we may write

$$\mathcal{R}\left(a \wedge b\right) = \mathcal{P}\left(S\left(b\right) \times S\left(a\right)\right) \tag{5.82}$$

or

$$\mathcal{R}\left(e_i \wedge e_j\right) = \mathcal{P}\left(S\left(e_j\right) \times S\left(e_i\right)\right) \tag{5.83}$$

Since both $S(a)$ and $S(b)$ are bivectors we can use equation B.13 (Appendix A) to reduce $S(b) \times S(a)$

$$\begin{aligned} S(b) \times S(a) &= \left(e^k \wedge \mathcal{P}_\perp (b \cdot \partial e_k)\right) \times \left(e^l \wedge \mathcal{P}_\perp (a \cdot \partial e_l)\right) \\ &= \left(e^k \cdot \mathcal{P}_\perp (a \cdot \partial e_l)\right) \mathcal{P}_\perp (b \cdot \partial e_k) \wedge e^l - \left(e^k \cdot e^l\right) \mathcal{P}_\perp (b \cdot \partial e_k) \wedge \mathcal{P}_\perp (a \cdot \partial e_l) \\ &\quad + \left(\mathcal{P}_\perp (b \cdot \partial e_k) \cdot e^l\right) e^k \wedge \mathcal{P}_\perp (a \cdot \partial e_l) - \left(\mathcal{P}_\perp (b \cdot \partial e_k) \cdot \mathcal{P}_\perp (a \cdot \partial e_l)\right) e^k \wedge e^l \end{aligned} \tag{5.84}$$

In equation 5.84 the first and third terms are zero. The second term is entirely outside the tangent space and the fourth term is entirely inside the tangent space. Also note that since the second term consists of bivectors that are entirely outside the tangent space that term commutes with all multivectors A in the tangent space so that the commutator of the second term with A is zero. Thus the Riemann tensor reduces to

$$\mathcal{R}(a \wedge b) = -\left(\mathcal{P}_\perp (b \cdot \partial e_u) \cdot \mathcal{P}_\perp (a \cdot \partial e_v)\right) e^u \wedge e^v \tag{5.85}$$

or

$$\mathcal{R}(e_i \wedge e_j) = -\left(\mathcal{P}_\perp (\partial_j e_u) \cdot \mathcal{P}_\perp (\partial_i e_v)\right) e^u \wedge e^v \tag{5.86}$$

To calculate the Riemann tensor in terms of the Christoffel symbols note that

$$[D_i, D_j] e_k = (S_j \times S_i) \times e_k \tag{5.87}$$

$$= (S_j \times S_i) \cdot e_k \tag{5.88}$$

$$= \mathcal{R}(e_i \wedge e_j) \cdot e_k \tag{5.89}$$

Equation 5.87 comes from letting $A = e_k$ in equation 5.80. Equation 5.88 comes from the fact that $S_j \times S_i$ is a bivector and that the commutator product of a bivector and a vector is the same as the dot product. Finally we have that $(S_j \times S_i) \cdot e_k = \mathcal{P}(S_j \times S_i) \cdot e_k$ since from equation 5.84 $\mathcal{P}_\perp (S_j \times S_i) = -\left(e^m \cdot e^l\right) \mathcal{P}_\perp (e_j \cdot \partial e_m) \wedge \mathcal{P}_\perp (e_i \cdot \partial e_l)$ so that $\mathcal{P}_\perp (S_j \times S_i) \cdot e_k = 0$. Thus

$$\begin{aligned} \mathcal{R}(e_i \wedge e_j) \cdot e_k &= [D_i, D_j] e_k \\ &= D_i \left(\Gamma^a_{jk} e_a\right) - D_j \left(\Gamma^a_{ik} e_a\right) \\ &= \left(\partial_i \Gamma^a_{jk}\right) e_a + \Gamma^a_{jk} D_i e_a - \left(\partial_j \Gamma^a_{ik}\right) e_a - \Gamma^a_{ik} D_j e_a \\ &= \left(\partial_i \Gamma^a_{jk} + \Gamma^b_{jk} \Gamma^a_{ib}\right) e_a - \left(\partial_j \Gamma^a_{ik} - \Gamma^b_{ik} \Gamma^a_{jb}\right) e_a \end{aligned} \tag{5.90}$$

so

$$\begin{aligned} \left(\mathcal{R}(e_i \wedge e_j) \cdot e_k\right) \cdot e^l &= \left(\left(\partial_i \Gamma^a_{jk}\right) + \Gamma^b_{jk} \Gamma^a_{ib}\right) \delta^l_a - \left(\left(\partial_j \Gamma^a_{ik}\right) + \Gamma^b_{ik} \Gamma^a_{jb}\right) \delta^l_a \\ &= \partial_i \Gamma^l_{jk} + \Gamma^b_{jk} \Gamma^l_{ib} - \partial_j \Gamma^l_{ik} - \Gamma^b_{ik} \Gamma^l_{jb} \end{aligned} \tag{5.91}$$

Using equation 5.86 we have

$$\left(\mathcal{R}\left(e_i\wedge e_j\right)\cdot e_k\right)\cdot e^l=-\left(\mathcal{P}_\perp\left(\partial_j e_u\right)\cdot\mathcal{P}_\perp\left(\partial_i e_v\right)\right)\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l \tag{5.92}$$

Using equation B.10 (Appendix A) to reduce $\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l$ gives

$$\left(\left(e^u\wedge e^v\right)\cdot e_k\right)\cdot e^l=g^{ul}\delta^v_k-g^{vl}\delta^u_k \tag{5.93}$$

Substituting equation 5.93 into equation 5.92 gives

$$\begin{aligned}\left(\mathcal{R}\left(e_i\wedge e_j\right)\cdot e_k\right)\cdot e^l&=\mathcal{P}_\perp\left(\partial_j e_k\right)\cdot\mathcal{P}_\perp\left(\partial_i e_v\right)g^{vl}-\mathcal{P}_\perp\left(\partial_j e_u\right)\cdot\mathcal{P}_\perp\left(\partial_i e_k\right)g^{ul}\\&=\mathcal{P}_\perp\left(\partial_j e_k\right)\cdot\mathcal{P}_\perp\left(\partial_i e^l\right)-\mathcal{P}_\perp\left(\partial_j e^l\right)\cdot\mathcal{P}_\perp\left(\partial_i e_k\right)\end{aligned} \tag{5.94}$$

because

$$\begin{aligned}\mathcal{P}_\perp\left(\partial_i e_v\right)g^{vl}&=\mathcal{P}_\perp\left(g^{vl}\partial_i e_v\right)\\&=\mathcal{P}_\perp\left(\partial_i\left(g^{vl}e_v\right)-\left(\partial_i g^{vl}\right)e_v\right)\\&=\mathcal{P}_\perp\left(\partial_i\left(g^{vl}e_v\right)\right)\\&=\mathcal{P}_\perp\left(\partial_i e^l\right)\end{aligned} \tag{5.95}$$

Finally

$$\begin{aligned}R_{ijk}{}^{l}&=\mathcal{P}_\perp\left(\partial_j e_k\right)\cdot\mathcal{P}_\perp\left(\partial_i e^l\right)-\mathcal{P}_\perp\left(\partial_j e^l\right)\cdot\mathcal{P}_\perp\left(\partial_i e_k\right)\\&=\partial_i\Gamma^l_{jk}+\Gamma^b_{jk}\Gamma^l_{ib}-\partial_j\Gamma^l_{ik}-\Gamma^b_{ik}\Gamma^l_{jb}\end{aligned} \tag{5.96}$$

Which is the standard form of the Riemann tensor in terms of the Christoffel symbols. Note that

$$\begin{aligned}R_{ijkl}&=R_{ijk}{}^{v}g_{vl}\\&=\mathcal{P}_\perp\left(\partial_j e_k\right)\cdot\mathcal{P}_\perp\left(\partial_i e_l\right)-\mathcal{P}_\perp\left(\partial_j e_l\right)\cdot\mathcal{P}_\perp\left(\partial_i e_k\right)\end{aligned} \tag{5.97}$$

From equation 5.97 and equation 5.53 ($\partial_i e_j=\partial_j e_i$) we can see that the symmetries of the covariant Riemann tensor are

$$R_{ijkl}=-R_{jikl},\ \ R_{ijkl}=-R_{ijlk},\ \text{ and } R_{ijkl}=R_{klij}$$

To prove the first Bianchi identity form $\mathcal{R}\left(e_i\wedge e_j\right)\cdot e_k$ and use equation 5.54 ($D_j e_k=D_k e_j$) to get

$$\begin{aligned}\mathcal{R}\left(e_i\wedge e_j\right)\cdot e_k&=D_iD_je_k-D_jD_ie_k\\&=D_iD_ke_j-D_jD_ke_i\\&=\left[D_i,D_k\right]e_j-\left[D_j,D_k\right]e_i+D_k\left(D_ie_j-D_je_i\right)\\&=\mathcal{R}\left(e_i\wedge e_k\right)\cdot e_j-\mathcal{R}\left(e_j\wedge e_k\right)\cdot e_i\end{aligned} \tag{5.98}$$

now defining the function $F(a,b,c)$ by

$$F(a,b,c) \equiv a\cdot\mathcal{R}(b\wedge c) + c\cdot\mathcal{R}(a\wedge b) + b\cdot\mathcal{R}(c\wedge a) = 0 \tag{5.99}$$

However $F(a,b,c)$ is a linear function of a, b, and c. Also $F(b,a,c) = -F(a,b,c)$ and $F(a,c,b) = -F(a,b,c)$ so since F is antisymmetric in all arguments we may write

$$F(a,b,c) = F(a\wedge b\wedge c) \tag{5.100}$$

Thus equation 5.99 contains $n\binom{n}{3} = \dfrac{n^2(n-1)(n-2)}{6}$ scalar coefficients. Since the Riemann tensor is a bivector valued function of a bivector the degrees of freedom of the tensor is no more than $\left(\dfrac{n(n-1)}{2}\right)^2$ and equation 5.99 reduces the degrees of freedom by $n\binom{n}{3}$ so that the total degrees of freedom of the Riemann tensor is

$$\left(\frac{n(n-1)}{2}\right)^2 - n\binom{n}{3} = \frac{1}{12}n^2\left(n^2-1\right).$$

5.4 Manifold Mappings

One way of illuminating the connection between geometric calculus on manifolds and the standard presentation of differential geometry is to study the effects of mappings from one manifold to another (including mapping of the manifold onto itself). Let $f:\mathcal{M}\rightarrow\mathcal{M}'$ define a mapping from the manifold $\mathcal{M}$ to $\mathcal{M}'$. For our purposes f is a diffeomorphism. That is f and all of its derivatives are continuous and invertible. Thus we can show that the tangent spaces of $\mathcal{M}$ and $\mathcal{M}'$ are of the same dimension. We shall denote the image of $x\in\mathcal{M}$ as x' so that $f(x) = x'$. Then if $x(\lambda)$ defines a curve on $\mathcal{M}$, then $f(x(\lambda)) = x'(\lambda)$ defines a curve on $\mathcal{M}'$. In summary -

$\mathcal{M}$: Manifold embedded in vector space V ($\mathcal{M}\subset V$)
s^i: Coordinates of manifold $\mathcal{M}$ such that $x(s^1,\dots,s^r)\in\mathcal{M}\subset V$, $\dim(V) > r$
u_i: Basis vectors for embedding space V ($x = x^iu_i\in\mathcal{M}$)
$x(\lambda)$: Trajectory in $\mathcal{M}$ such that $x(\lambda) = x(s^1(\lambda),\dots,s^r(\lambda))$
$\mathcal{M}'$: Manifold embedded in vector space V'
u_i': Basis vectors for V'
f: Diffeomorphism from V to V'
x': Element of $\mathcal{M}'$ as an image of $x\in\mathcal{M}$ ($x' = f(x)\in\mathcal{M}'\subset V'$)
f^i: Component of f in V' such that $f(x) = f^i(x)\,u_i'$

Since $\dfrac{dx}{d\lambda}$ is in the tangent space of $\mathcal{M}$ and (remember that the dot product comes before the geometric product)

$$\begin{aligned}
\frac{dx'}{d\lambda} &= \frac{d}{d\lambda}\left(f\left(x\left(\lambda\right)\right)\right) \\
&= \left(\frac{dx}{d\lambda}\cdot\partial\right) f\left(x\right) \\
&= \left(\frac{ds^i}{d\lambda}\frac{\partial x}{\partial s^i}\cdot e^k\partial_k\right)\left(f^j\left(x\right)u'_j\right) \\
&= \left(\frac{ds^i}{d\lambda}e_i\cdot e^k\partial_k\right)\left(f^j\left(x\right)u'_j\right) \\
&= \left(\frac{ds^i}{d\lambda}\delta_i^k\partial_k\right)\left(f^j\left(x\right)u'_j\right) \\
&= \frac{ds^i}{d\lambda}\frac{\partial f^j}{\partial s^i}u'_j
\end{aligned} \tag{5.101}$$

is in the tangent space of $\mathcal{M}'$. Thus if a is in the tangent space of $\mathcal{M}$ at point x then

$$a' = \left(a\cdot\partial\right) f\left(x\right) = \mathsf{f}\left(a;x\right) = \mathsf{f}\left(a\right) \tag{5.102}$$

is in the tangent space of $\mathcal{M}'$ at point x' and the frame (basis) vectors for $\mathcal{M}$ at point x map from one tangent space to the other via

$$e'_i = \left(e_i\cdot\partial\right) f\left(x\right) = \mathsf{f}\left(e_i;x\right) = \mathsf{f}\left(e_i\right) \tag{5.103}$$

where in the rhs of equations 5.102 and 5.103 we suppress the position dependence, x, in the linear functional f. Since $f\left(x\right)$ is invertible for all derivatives there is no e_i such that $\mathsf{f}\left(e_i\right) = 0$ (the dimension of the image tangent space is the same as the original). Note that we actually have $e_i\left(x\right)$ and $e'_i\left(x\right)$. An example is shown in figure 5.1

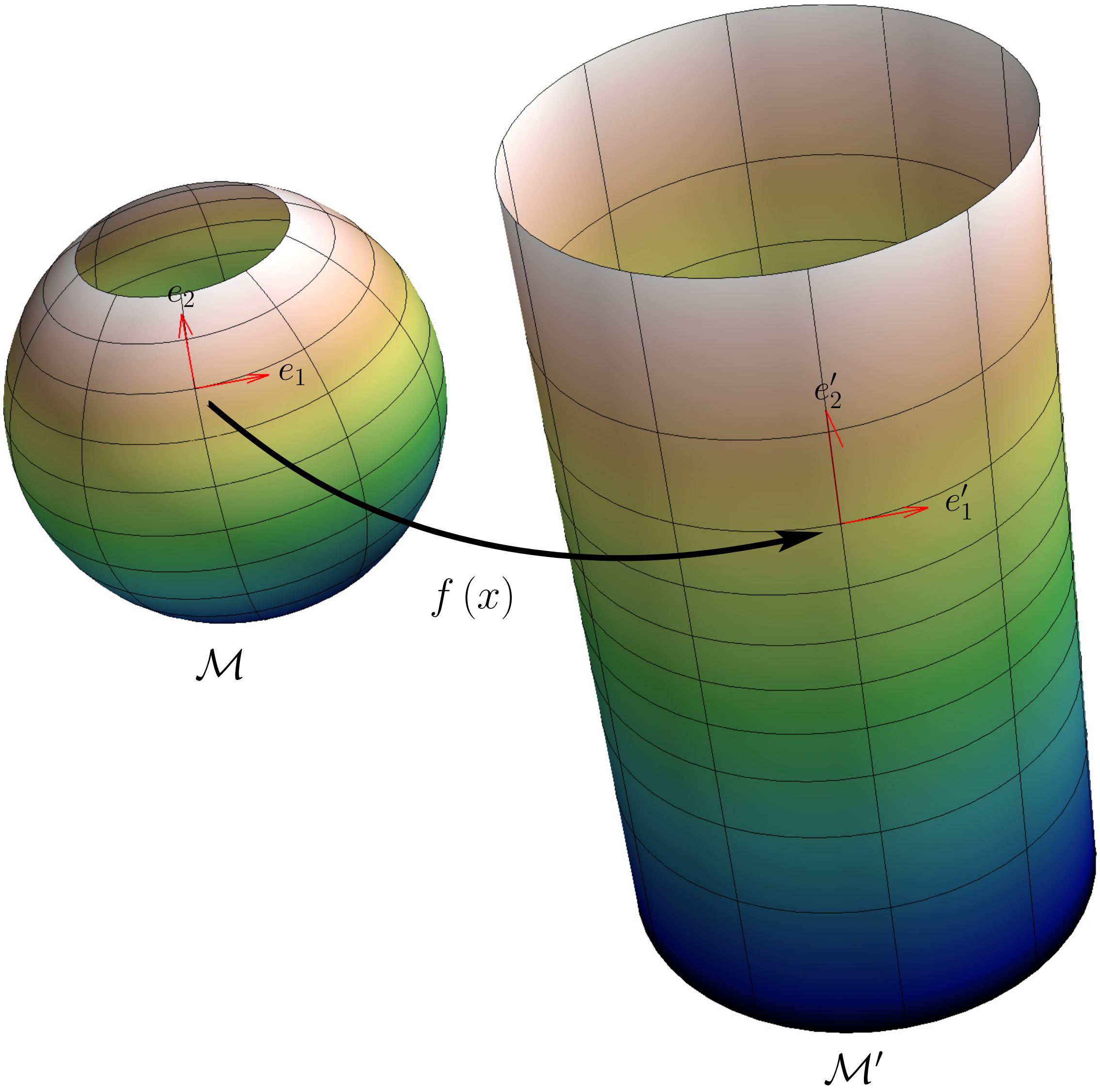
The cotangent frame, e^i, in $\mathcal{M}$ is mapped to e'^i in $\mathcal{M}'$ via the adjoint of the inverse

$$e'^i = \overline{\mathsf{f}^{-1}}\left(e^i\right). \tag{5.104}$$

This is simply proved by noting (remember that by definition $a\cdot\overline{\mathsf{f}}\left(b\right) = b\cdot\mathsf{f}\left(a\right)$)

$$\begin{aligned}
e'_i\cdot e'^j &= \mathsf{f}\left(e_i\right)\cdot\overline{\mathsf{f}^{-1}}\left(e^j\right) \\
&= e^j\cdot\mathsf{f}^{-1}\left(\mathsf{f}\left(e_i\right)\right) \\
&= e^j\cdot e_i \\
&= \delta_i^j
\end{aligned} \tag{5.105}$$

$$\mathcal{M}:\left\{x=\cos\left(s^{2}\right)\left(\cos\left(s^{1}\right)u_{1}+\sin\left(s^{1}\right)u_{2}\right)+\sin\left(s^{2}\right)u_{3}:\left|s^{2}\right|\leq s^{2}_{max}<\frac{\pi}{2}\right\}$$



$$\mathcal{M}':\left\{f\left(x\right)=\cos\left(s^{1}\right)u_{1}'+\sin\left(s^{1}\right)u_{2}'+\tan\left(s^{2}\right)u_{3}':\left|s^{2}\right|\leq s^{2}_{max}<\frac{\pi}{2}\right\}$$

$$e_{1}'\left(x\right)=\mathsf{f}\left(e_{1}\right)=-\sin\left(s^{1}\right)u_{1}'+\cos\left(s^{1}\right)u_{2}'$$

$$e_{2}'\left(x\right)=\mathsf{f}\left(e_{2}\right)=\frac{u_{3}'}{\cos^{2}\left(s^{2}\right)}$$

Figure 5.1: Mercator mapping of sphere to cylinder manifold. Top and bottom of sphere excluded.

The exterior product of two tangent vectors, $e_i \wedge e_j$, maps to

$$e_i \wedge e_j \mapsto e'_i \wedge e'_j = \mathsf{f}\left(e_i \wedge e_j\right) \tag{5.106}$$

since f is linear in tangent space arguments. Likewise if I' is the "unit" pseudoscalar for the tangent space of $\mathcal{M}'$ at $f(x)$ ($I'^2 = \pm 1$) and I the "unit" pseudoscalar for the tangent space of $\mathcal{M}$ at x ($I^2 = \pm 1$) then

$$\mathsf{f}(I) = \det(\mathsf{f})\, I'. \tag{5.107}$$

Since f is invertible (the tangent space and its image are isomorphic) we can apply the definition of determinant in equation 1.79 to equation 5.107. For cotangent vectors e^i and e^j

$$e'^i \wedge e'^j \mapsto \overline{\mathsf{f}^{-1}}\left(e^i\right) \wedge \overline{\mathsf{f}^{-1}}\left(e^j\right) = \overline{\mathsf{f}^{-1}}\left(e^i \wedge e^j\right). \tag{5.108}$$

Since the derivative of a scalar field, ϕ, is a cotangent vector $\partial\phi = e^i \partial_i \phi$ and $\phi(x) = \phi'(x')$ we can write

$$\partial' = \overline{\mathsf{f}^{-1}}(\partial) = \overline{\mathsf{f}^{-1}}\left(e^i \partial_i\right) = e'^i \partial_i. \tag{5.109}$$

Since D is also a cotangent vector we also have

$$D' = \overline{\mathsf{f}^{-1}}(D). \tag{5.110}$$

For the directional derivative of a scalar field

$$\begin{aligned}
\left(a' \cdot \partial'\right)\phi' &= \left(\mathsf{f}(a) \cdot \overline{\mathsf{f}^{-1}}\left(\dot{\partial}\right)\right)\dot{\phi} \\
&= \left(\dot{\partial} \cdot \mathsf{f}^{-1}\left(\mathsf{f}(a)\right)\right)\dot{\phi} \\
&= \left(\dot{\partial} \cdot a\right)\dot{\phi} \\
&= (a \cdot \partial)\,\phi
\end{aligned} \tag{5.111}$$

or for a vector field

$$\left(a' \cdot \partial'\right) b' = (a \cdot \partial)\,\mathsf{f}(b) \tag{5.112}$$

The covariant derivative is constructed using the projection operator, $\mathcal{P}$, which contains a contraction with the pseudoscalar I. Thus the covariant derivative depends upon the metric encoded by $I(x)$.

Consider the following operation where a and b are tangent vectors (from now on the parenthesis around the dot operands are implied) and using equations 5.49 and 5.50 we get

$$\begin{aligned}
a \cdot \partial b - b \cdot \partial a &= a \cdot Db - b \cdot Da - a \cdot S(b) + b \cdot S(a) \\
&= a \cdot Db - b \cdot Da
\end{aligned} \tag{5.113}$$

since $a\cdot S\left(b\right) = b\cdot S\left(a\right)$ by equation 5.56. Now define the Lie derivative of a with respect to b by

$$\mathcal{L}_a b \equiv a\cdot\partial b - b\cdot\partial a \tag{5.114}$$

We will show that

$$\mathcal{L}_a b \mapsto \mathcal{L}'_{a'}b' = \mathsf{f}\left(\mathcal{L}_a b\right). \tag{5.115}$$

First note that

$$\left(a\cdot\partial\right)\mathsf{f}\left(b\right) - \left(b\cdot\partial\right)\mathsf{f}\left(a\right) = \mathsf{f}\left(\left(a\cdot\partial\right)b - \left(b\cdot\partial\right)a\right) + \left(a\cdot\dot{\partial}\right)\dot{\mathsf{f}}\left(b\right) - \left(b\cdot\dot{\partial}\right)\dot{\mathsf{f}}\left(a\right) \tag{5.116}$$

Note that since $\mathsf{f}\left(a\right)$ is the differential of $f\left(x\right)$ we have $\left(\partial_i e_j - \partial_j e_i = 0\right)$

$$\begin{aligned} 0 = \left(\partial_i\partial_j - \partial_j\partial_i\right)f\left(x\right) &= \partial_i\mathsf{f}\left(e_j\right) - \partial_j\mathsf{f}\left(e_i\right) \\ 0 &= \dot{\partial}_i\dot{\mathsf{f}}\left(e_j\right) - \dot{\partial}_j\dot{\mathsf{f}}\left(e_i\right) + \mathsf{f}\left(\partial_i e_j - \partial_j e_i\right) \\ 0 &= \partial_i e'_j - \partial_j e'_i \end{aligned} \tag{5.117}$$

so that $\partial_i e'_j - \partial_j e'_i = 0$ and $\dot{\partial}_i\dot{\mathsf{f}}\left(e_j\right) - \dot{\partial}_j\dot{\mathsf{f}}\left(e_i\right) = 0$. Thus

$$\begin{aligned} \left(a\cdot\dot{\partial}\right)\dot{\mathsf{f}}\left(b\right) - \left(b\cdot\dot{\partial}\right)\dot{\mathsf{f}}\left(a\right) &= \left(a^k e_k\cdot e^i\dot{\partial}_i\right)\dot{\mathsf{f}}\left(b^j e_j\right) - \left(b^k e_k\cdot e^j\dot{\partial}_j\right)\dot{\mathsf{f}}\left(a^i e_i\right) \\ &= a^i\dot{\partial}_i\dot{\mathsf{f}}\left(b^j e_j\right) - b^j\dot{\partial}_j\dot{\mathsf{f}}\left(a^i e_i\right) \\ &= a^i b^j\dot{\partial}_i\dot{\mathsf{f}}\left(e_j\right) - b^j a^i\dot{\partial}_j\dot{\mathsf{f}}\left(e_i\right) \\ &= a^i b^j\left(\dot{\partial}_i\dot{\mathsf{f}}\left(e_j\right) - \dot{\partial}_j\dot{\mathsf{f}}\left(e_i\right)\right) = 0 \end{aligned} \tag{5.118}$$

and

$$\left(a\cdot\partial\right)\mathsf{f}\left(b\right) - \left(b\cdot\partial\right)\mathsf{f}\left(a\right) = \mathsf{f}\left(\left(a\cdot\partial\right)b - \left(b\cdot\partial\right)a\right) \tag{5.119}$$

and by equations 5.112 and 5.119 we have

$$\mathcal{L}'_{a'}b' = \mathsf{f}\left(\mathcal{L}_a b\right) \tag{5.120}$$

and the Lie derivative maps simply under f.

Since $e'^k\cdot e'_j = \delta^k_j$ and $e'^k\cdot e'_i = \delta^k_i$ we have (using equation 5.117)

$$\begin{aligned} \partial_i\left(e'^k\cdot e'_j\right) - \partial_j\left(e'^k\cdot e'_i\right) &= 0 \\ e'^k\cdot\left(\partial_i e'_j - \partial_j e'_i\right) + e'_j\cdot\partial_i e'^k - e'_i\cdot\partial_j e'^k &= 0 \\ e'_j\cdot\partial_i e'^k - e'_i\cdot\partial_j e'^k &= 0 \\ \mathsf{f}\left(e_j\right)\cdot\partial_i\overline{\mathsf{f}^{-1}}\left(e^k\right) - \mathsf{f}\left(e_i\right)\cdot\partial_j\overline{\mathsf{f}^{-1}}\left(e^k\right) &= 0 \end{aligned} \tag{5.121}$$

Equation 5.121 inplies that $\mathcal{P}'\left(\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)\right)=0$ as can be shown by expanding

$$\begin{aligned}\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)&=\overline{\mathsf{f}^{-1}}\left(e^i\partial_i\right)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)\\&=e'^i\partial_i\wedge e'^k\end{aligned}\tag{5.122}$$

Since equation 5.122 is a grade two multivector we can expand it in terms of the blades (components) $e'^l\wedge e'^m$ as follows from equations 1.75, 1.76, B.10, and the fact that ∂_i is a scalar operator that commutes with $\wedge$.

$$\begin{aligned}\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)&=e'^i\partial_i\wedge e'^k\\\mathcal{P}'\left(\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)\right)&=\left(\left(e'_m\wedge e'_l\right)\cdot\left(e'^i\partial_i\wedge e'^k\right)\right)\left(e'^l\wedge e'^m\right)\\&=\left(\left(e'_m\wedge e'_l\right)\cdot\left(e'^i\wedge\partial_ie'^k\right)\right)\left(e'^l\wedge e'^m\right)\\&=\left(\left(e'_m\cdot\partial_ie'^k\right)\left(e'_l\cdot e'^i\right)\right)-\left(\left(e'_m\cdot e'^i\right)\left(e'_l\cdot\partial_ie'^k\right)\right)\left(e'^l\wedge e'^m\right)\\&=\left(\delta^i_l\left(e'_m\cdot\partial_ie'^k\right)-\delta^i_m\left(e'_l\cdot\partial_ie'^k\right)\right)\left(e'^l\wedge e'^m\right)\\&=\left(\left(e'_m\cdot\partial_le'^k\right)-\left(e'_l\cdot\partial_me'^k\right)\right)\left(e'^l\wedge e'^m\right)\end{aligned}\tag{5.123}$$

But the coefficient of $e'^l\wedge e'^m$ is the same as equation 5.121 so that $\mathcal{P}'\left(\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)\right)=0$. Note that since we expanded $\overline{\mathsf{f}^{-1}}(\partial)\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)$ in terms of $e'^l\wedge e'^m$ (cotangent vectors in $\mathcal{M}'$) it was automatically projected into the tangent space of $\mathcal{M}'$ at $x'=f(x)$.

Since $D'=\mathcal{P}'\left(\overline{\mathsf{f}^{-1}}(\partial)\right)$ we have

$$D'\wedge e'^k=D'\wedge\overline{\mathsf{f}^{-1}}\left(e^k\right)=0.\tag{5.124}$$

and using the product rule the exterior derivative of a blade formed from cotangent vectors is (just move the term in the product being differentiated to the front using the alternating property of the "$\wedge$" product)

$$D'\wedge\left(e'^{k_1}\wedge\ldots\wedge e'^{k_r}\right)=0.\tag{5.125}$$

Now expand a grade-r multivector field $A(x)$ on $\mathcal{M}$ in terms of the cotangent vectors

$$A(x)=A_{i_1i_2\ldots i_r}(x)\,e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_r}\tag{5.126}$$

Then

$$D\wedge A(x)=\left(DA_{i_1i_2\ldots i_r}(x)\right)\wedge e^{i_1}\wedge e^{i_2}\wedge\ldots\wedge e^{i_r}\tag{5.127}$$

since the exterior derivate of a scalar field is the same as the covariant derivative and that the exterior derivative of a basis blade is zero. Likewise

$$D'\wedge A'(x)=\left(D'A_{i_1i_2\ldots i_r}(x)\right)\wedge e'^{i_1}\wedge e'^{i_2}\wedge\ldots\wedge e'^{i_r}\tag{5.128}$$

so that for a general multivector field $A(x)$ on $\mathcal{M}$ we have

$$D \wedge A \mapsto D' \wedge A' = \overline{\mathsf{f}^{-1}}\,(D \wedge A)\,. \tag{5.129}$$

5.5 The Fundmental Theorem of Geometric Calculus on Manifolds

The fundamental theorem of geometric calculus is simply implemented on manifolds. In figure 5.2 a simplicial decomposition of a surface defined by a close curve on a spherical manifold is shown. The only difference between the derivation of the fundamental theorem of geometical calculus on a vector space (which we have proved) and on a manifold is that the pseudo-scalar is not a constant but is now a function of position, $I(x)$, on the manifold. If we are not on a manifold the pseudo scalars corresponding to the oriented volume of each simplex are proportional to one another (they are equal when normalized) so have the same orientation. For a manifold the orientation of $I(x)$ can change with the position vector x. In the case of figure 5.2 $I(x)$ is a bi-vector defined by the tangent space (tangent plane) for each point on the sphere.

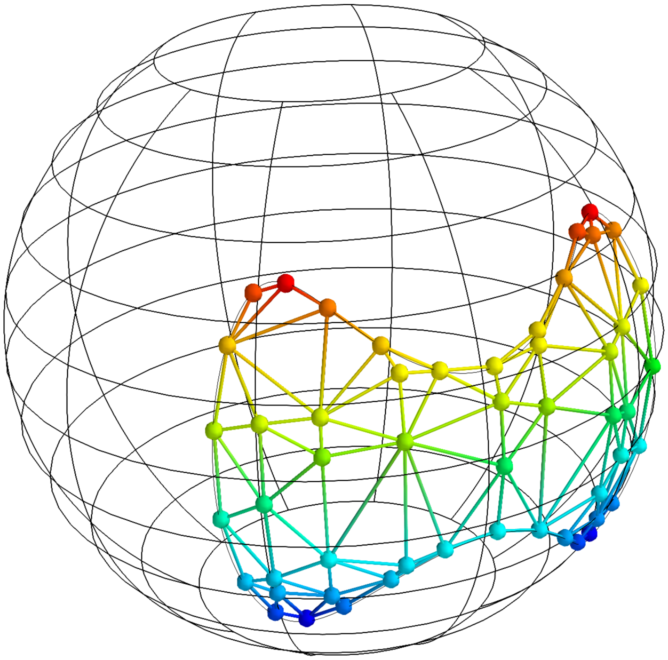


Figure 5.2: Simplical decomposition for surface defined by closed curve on spherical manifold.

Now consider the directed volume element (in the case of figure 5.2 an area element) for each simplex in figure 5.2 given by $\Delta X = \frac{1}{k!} e_1 \wedge \ldots \wedge e_k$ ($k = 2$ for figure 5.2). As the volume (area) of $\Delta X \rightarrow 0$, $\Delta X \propto I(x)$.

In equation 4.75 where on the l.h.s. of the equation we are integrating over the boundary of the simplex where f is a linear approximation to an arbitrary multivector function.

$$\oint_{\partial(x)_{(k)}} f(x) = \dot{f}\dot{\nabla} \cdot (\Delta X)$$

consider the operator $\dot{\nabla} \cdot (\Delta X)$ where we have left the dot on the ∇ to emphasize that it is not differentiating the ΔX. On a given simplex

$$\nabla = e^i \frac{\partial}{\partial \lambda^i} \tag{5.130}$$

where the λ^i's are the simplical coordinates (equation 4.19) and the e^i's are the reciprocal vectors to the e_i's that define the simplex. In the case of a manifold as the volume of $\Delta X \rightarrow 0$ the e_i's (since the e^i's define the same subspace as the e_i's) define a pseudoscalar that is proportional to $I(x)$ so that $e^i \frac{\partial}{\partial \lambda^i}$ is the projection of the geometric derivative, ∇, from the embedding vector space of the manifold to the tangent space of the manifold at point x. Thus in the case of a manifold as the volume of $\Delta X \rightarrow 0$ we have $\partial = e^i \frac{\partial}{\partial \lambda^i}$.

Note that

$$\dot{\partial}(\Delta X) = \dot{\partial} \cdot (\Delta X) + \dot{\partial} \wedge (\Delta X) \tag{5.131}$$

but $\dot{\partial} \wedge (\Delta X) = 0$ since ∂ is within the subspace (tangent space) defined by ΔX. The fundamental theorem of geometric calculus as applied to a manifold is

$$\oint_{\partial V} \mathsf{L}(dS) = \int_V \dot{\mathsf{L}}\left(\dot{\nabla} \cdot dX\right) = \int_V \dot{\mathsf{L}}\left(\dot{\partial} dX\right) \tag{5.132}$$

One can write $dX = I(x)|dX|$ and note that ∂ or ∇ in equation 5.132 do not differentiate $I(x)$.

5.5.1 Divergence Theorem on Manifolds

Let $\mathsf{L}(A) = \langle JAI^{-1} \rangle$ where J is a vector field in the tangent space of the manifold and substitute into equation 5.132 to get

$$\oint_{\partial V} \langle JdSI^{-1} \rangle = \int_V \left(\left\langle \dot{J}\dot{\partial} dXI^{-1} \right\rangle + \left\langle J\dot{\partial} dX\dot{I}^{-1} \right\rangle \right). \tag{5.133}$$

Now reduce equation 5.133 by noting that $ndS = I\,|dS|$ where n is the outward normal to the surface element dS. We can define an outward normal in n-dimensional manifold since the grade of dS is $n-1$ and it defines a subspace of the tangent space of dimension $n-1$ for which a unique normal exists. This gives

$$\begin{aligned}
dS &= \frac{n}{n^2} I\,|dS| && (5.134)\\
\left\langle JdSI^{-1}\right\rangle &= \frac{|dS|}{n^2}\left\langle JnII^{-1}\right\rangle &&\\
&= \frac{|dS|}{n^2}\left\langle Jn\right\rangle &&\\
&= \frac{J\cdot n\,|dS|}{n^2} && (5.135)
\end{aligned}$$

Also

$$\begin{aligned}
\dot{J}\dot{\partial}dXI^{-1} &= \dot{J}\dot{\partial}I\,|dX|\,I^{-1} &&\\
&= \dot{J}\dot{\partial}\,|dX| && (5.136)\\
\left\langle \dot{J}\dot{\partial}dXI^{-1}\right\rangle &= \left\langle \dot{J}\dot{\partial}\,|dX|\right\rangle &&\\
&= \left\langle \dot{J}\dot{\partial}\right\rangle |dX| &&\\
&= \partial\cdot J\,|dX|\,. && (5.137)
\end{aligned}$$

Finally, since both I^{-1} and dX are proportional to I

$$\begin{aligned}
\left\langle J\dot{\partial}\dot{I}^{-1}dX\right\rangle &= \pm\left\langle J\dot{\partial}\dot{I}I\right\rangle |dX| &&\\
&= \pm\frac{1}{2}\left\langle J\dot{\partial}\left(\dot{I}I + I\dot{I}\right)\right\rangle |dX| &&\\
&= \pm\frac{1}{2}\left\langle J\partial\left(I^2\right)\right\rangle |dX| &&\\
&= 0 && (5.138)
\end{aligned}$$

so that the divergence theorem is

$$\oint_{\partial V} \frac{n}{n^2}\cdot J\,|dS| = \int_V \partial\cdot J\,|dX| \tag{5.139}$$

5.5.2 Stokes Theorem on Manifolds

Assume that the manifold tangent space dimension is $s+1$ and B_r is a grade r multivector field. For clarity we denote the grade of volume and surface elements with a subscript on the

d's so that $dX = d_{s+1}X$ and $dS = d_sS$. Now let $\mathsf{L}(A) = \left\langle B_rA\right\rangle_{|s-r|}$ so that the application of equation 5.132 gives

$$
\begin{aligned}
\oint_{\partial V}\left\langle B_rd_sS\right\rangle_{|s-r|} &= \int_V\left\langle \dot{B}_r\dot{\partial}d_{s+1}X\right\rangle_{|s-r|} \\
\oint_{\partial V}\overbrace{B_r\cdot d_sS}^{\text{grade }|s-r|} &= \int_V\left\langle\left(\overbrace{\dot{B}_r\wedge\dot{\partial}}^{\text{grade }r+1} + \underbrace{\dot{B}_r\cdot\dot{\partial}}_{\text{grade }r-1}\right)d_{s+1}X\right\rangle_{|s-r|} \\
&= (-1)^r\int_V\left\langle\underbrace{\left(\partial\wedge B_r\right)d_{s+1}X}_{\text{lowest grade is }|s-r|}\right\rangle_{|s-r|} + \int_V\left\langle\underbrace{\left(\dot{B}_r\cdot\dot{\partial}\right)d_{s+1}X}_{\text{lowest grade is }|s-r+2|}\right\rangle_{|s-r|} \\
&= (-1)^r\int_V\left(\partial\wedge B_r\right)\cdot d_{s+1}X \qquad (5.140)
\end{aligned}
$$

However

$$
\left(\partial\wedge B_r\right)\cdot d_{s+1}X = \left(D\wedge B_r\right)\cdot d_{s+1}X \tag{5.141}
$$

since $d_{s+1}X = I_{s+1}(x)\left|d_{s+1}X\right|$ and the dot product of any component of $\partial\wedge B_r$ that is not in the tangent space defined by $I_{s+1}(x)$ is zero so that

$$
\oint_{\partial V}B_r\cdot d_sS = \int_V\left(\dot{B}_r\wedge\dot{D}\right)\cdot d_{s+1}X = (-1)^r\int_V\left(D\wedge B_r\right)\cdot d_{s+1}X \tag{5.142}
$$

The divergence theorem is recovered when $r = s-1$. This is important for constructing conservation theorems in curved spaces. Equation 5.142 is the most general application of equation 5.132 that allows one to replace ∂ with D (covariant derivative) on a manifold.

5.6 Differential Forms in Geometric Calculus

5.6.1 Inner Products of Subspaces

Start by considering products of the form where A_r and B_r are r-grade blades

$$A_r^\dagger \cdot B_r = (a_1 \wedge \ldots \wedge a_r)^\dagger \cdot (b_1 \wedge \ldots \wedge b_r) \tag{5.143}$$

Both A_r and B_r define r-dimensional subspaces of the vector space via the equations $x \wedge A_r = 0$ and $x \wedge B_r = 0$. We show that if A_r and B_r do not define the same subspace then $A_r^\dagger \cdot B_r = 0$. Assume that they do not define the same subspaces, but that the intersection of the subspaces has dimension $s < r$ so that one can have an orthogonal basis for A_r of $\{e_1, \ldots, e_s, e_{s+1}, \ldots, e_r\}$ and an orthogonal basis for B_r of $\left\{e_1, \ldots, e_s, e'_{s+1}, \ldots, e'_r\right\}$. Then the geometric product of $A_r^\dagger$ and B_r is

$$\begin{aligned} A_r^\dagger B_r &= \alpha e_r \ldots e_{s+1} e_s \ldots e_1 \beta e_1 \ldots e_s e'_{s+1} \ldots e'_r \\ &= \left(\alpha\beta e_1^2 \ldots e_s^2\right) e_{s+1} \ldots e_1 e'_{s+1} \ldots e'_r \end{aligned} \tag{5.144}$$

where the quantity in parenthesis is a scalar and the other factors a blade of grade $2(r-s)$. Thus

$$A_r^\dagger \cdot B_r = \left\langle A_r^\dagger B_r \right\rangle = 0. \tag{5.145}$$

If A_r and B_r define the same r-dimensional subspace let $\{e_1, \ldots, e_r\}$ be an orthogonal basis for the subspace and expand A_r and B_r in terms of the orthogonal basis vectors and reciprocal orthogonal basis vectors respectively -

$$a_i = \left(a_i \cdot e^j\right) e_j \tag{5.146}$$

and

$$b_i = (b_i \cdot e_j)\, e^j \tag{5.147}$$

Let the matrices of the coefficients in equations 5.146 and 5.147 be denoted by $[a_i \cdot e^j]$ and $[b_i \cdot e_j]$. Then we can expand A_r and B_r as

$$A_r = \det\left(\left[a_i \cdot e^j\right]\right) e_1 \ldots e_r \tag{5.148}$$

and

$$B_r = \det\left([b_i \cdot e_j]\right) e^1 \ldots e^r \tag{5.149}$$

and (since the determinant of a matrix and determinant of the transpose of matrix are equal)

$$\begin{aligned}
A_r^\dagger\cdot B_r &= \det\left(\left[a_i\cdot e^j\right]\right)\det\left(\left[b_i\cdot e_j\right]\right)\left\langle e_r\dots e_1e^1\dots e^r\right\rangle\\
&= \det\left(\left[a_i\cdot e^j\right]\right)\det\left(\left[b_i\cdot e_j\right]\right)\\
&= \det\left(\left[a_i\cdot e^j\right]\right)\det\left(\left[b_i\cdot e_j\right]^T\right)\\
&= \det\left(\left[a_i\cdot e^j\right]\right)\det\left(\left[b_j\cdot e_i\right]\right)\\
&= \det\left(\left[\left(a_i\cdot e^k\right)\left(b_j\cdot e_k\right)\right]\right) \qquad (5.150)
\end{aligned}$$

But

$$\begin{aligned}
a_i\cdot b_j &= \left(a_i\cdot e^k\right)e_k\cdot\left(b_j\cdot e_l\right)e^l\\
&= \left(a_i\cdot e^k\right)\left(b_j\cdot e_l\right)\delta_k^l\\
&= \left(a_i\cdot e^k\right)\left(b_j\cdot e_k\right) \qquad (5.151)
\end{aligned}$$

So that

$$A_r^\dagger\cdot B_r = \det\left(\left[a_i\cdot b_j\right]\right). \qquad (5.152)$$

From our derivations we see that if A_r is a general r-grade multivector (not a blade) we can always find a r-grade blade such that

$$A_r^\dagger\cdot\left(b_1\wedge\dots\wedge b_r\right) = \left(a_1\wedge\dots\wedge a_r\right)^\dagger\cdot\left(b_1\wedge\dots\wedge b_r\right). \qquad (5.153)$$

Now consider the relation between the basis $\{a_1,\dots,a_n\}$ and the reciprocal basis $\{a^1,\dots,a^n\}$ for an n-dimensional vector space where in equation 5.154 $r\le n$ and $1\le i_l, j_m\le n$

$$\left(a_{i_1}\wedge\dots\wedge a_{i_r}\right)^\dagger\cdot\left(a^{j_1}\wedge\dots\wedge a^{j_r}\right) = \det\left(\left[a_{i_l}\cdot a^{j_m}\right]\right) = \det\left(\left[\delta_{i_l}^{j_m}\right]\right) \qquad (5.154)$$

where $i_1 < i_2 < \dots < i_r$ and $j_1 < j_2 < \dots < j_r$. Equation 5.154 is zero unless $i_l = j_l$ for all $1\le l\le r$ so that

$$\left(a_{i_1}\wedge\dots\wedge a_{i_r}\right)^\dagger\cdot\left(a^{j_1}\wedge\dots\wedge a^{j_r}\right) = \delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\dots\delta_{i_r}^{j_r}. \qquad (5.155)$$

since if the index ordering condition is satisfied for the i_l's and the j_m's and there is an index, l, such that $i_l\ne j_l$ then the two blades do not define the same subspace and the inner product is zero. The square of a blade is given by

$$\begin{aligned}
\left(a_1\wedge\dots\wedge a_r\right)^2 &= \left(a_1\wedge\dots\wedge a_r\right)\cdot\left(a_1\wedge\dots\wedge a_r\right) \qquad (5.156)\\
&= (-1)^{\frac{r(r-1)}{2}}\left(a_1\wedge\dots\wedge a_r\right)^\dagger\cdot\left(a_1\wedge\dots\wedge a_r\right) \qquad (5.157)\\
&= (-1)^{\frac{r(r-1)}{2}}\det\left(\left[a_i\cdot a_j\right]\right) \qquad (5.158)\\
&= (-1)^{\frac{r(r-1)}{2}}a_1^2\dots a_r^2 \text{ if the } a_i\text{'s are orthogonal.} \qquad (5.159)
\end{aligned}$$

5.6.2 Alternating Forms

If V is a vector space then an r-rank tensor is a multilinear map

$$T_r\left(v_1,\ldots,v_r\right):\bigotimes_{i=1}^{r}V\rightarrow\mathcal{R} \tag{5.160}$$

where ($\otimes$ is the cartesian product of vector spaces)

$$\bigotimes_{i=1}^{r}V=\underbrace{V\otimes\cdots\otimes V}_{r\text{ times}} \tag{5.161}$$

so that if $v_i\in V$ the tuple $\left(v_1,\ldots,v_r\right)\in\bigotimes\limits_{i=1}^{r}V$.

The sum of two r-rank tensors is the r-rank tensor defined by

$$\left(A_r+B_r\right)\left(v_1,\ldots,v_r\right)\equiv A_r\left(v_1,\ldots,v_r\right)+B_r\left(v_1,\ldots,v_r\right). \tag{5.162}$$

The tensor product of rank r and s tensors is a rank $r+s$ tensor defined by

$$\left(A_r\otimes B_s\right)\left(v_1,\ldots,v_{r+s}\right)\equiv A_r\left(v_1,\ldots,v_r\right)B_s\left(v_{r+1},\ldots,v_{r+s}\right). \tag{5.163}$$

An r-form (when we say something is a form from now on we mean alternating form) is an r-rank tensor with the property (no summation in this case)

$$\alpha_r\left(v_1,\ldots,v_r\right)=\epsilon^{i_1\ldots i_r}_{1\ldots r}\alpha_r\left(v_{i_1},\ldots,v_{i_r}\right) \tag{5.164}$$

where $\epsilon^{i_1\ldots i_r}_{1\ldots r}$ is the mixed rank permutation symbol. We can always construct an r-form from an r-rank tensor via

$$\alpha_r\left(v_1,\ldots,v_r\right)=\epsilon\left(A_r\right)\equiv\sum_{i_1,\ldots,i_r}\epsilon^{1\ldots r}_{i_1\ldots i_r}A_r\left(v_{i_1},\ldots,v_{i_r}\right) \tag{5.165}$$

In the geometric algebra a simple representation of an alternating r-form is

$$\alpha_r\left(v_1,\ldots,v_r\right)=A_r^{\dagger}\cdot\left(v_1\wedge\ldots\wedge v_r\right) \tag{5.166}$$

where A_r is a grade-r multivector. Since the grade of A_r and the grade of $\left(v_1\wedge\ldots\wedge v_r\right)$ are the same the inner product results in a scalar and also since $\left(v_1\wedge\ldots\wedge v_r\right)$ is a blade it alternates sign upon exchange of adjacent vectors.

The basic operations of the "Algebra of Forms" inherit the sum and cartesian product operations from tensor since they are tensors. However, in order to construct an algebra of forms we need a product of two forms that results in a form. The tensor product of two alternating forms is not an alternating form, but since we know how to convert a tensor to a form we define the exterior product of an r-form and s-form to be the $r+s$ form

$$\alpha_r \hat{\wedge} \beta_s \equiv \epsilon\left(\alpha_r \otimes \beta_s\right) \tag{5.167}$$

as an example

$$\alpha_1\left(v_1\right) \hat{\wedge} \beta_1\left(v_2\right) = \alpha_1\left(v_1\right)\beta_1\left(v_2\right) - \alpha_1\left(v_2\right)\beta_1\left(v_1\right) \tag{5.168}$$

so that

$$\alpha_r\left(v_1,\dots,v_r\right) \hat{\wedge} \beta_s\left(v_{r+1},\dots,v_{r+s}\right) = \left(A_r \wedge B_s\right)^\dagger \cdot \left(v_1 \wedge \dots \wedge v_{r+s}\right) \tag{5.169}$$

and

$$\alpha_r \hat{\wedge} \beta_s = (-1)^{rs}\, \beta_s \hat{\wedge} \alpha_r \tag{5.170}$$

The interior product of an r and s-form is defined by ($r > s$) an $r-s$ form (note that we are distinguishing the inner, $\boldsymbol{\cdot}$, and exterior, $\hat{\wedge}$, products for forms from the geometric algebra products by using boldface symbols)

$$\beta_s \boldsymbol{\cdot} \alpha_r \equiv \left(B_s \cdot A_r\right)^\dagger \cdot \left(v_{s+1} \wedge \dots \wedge v_r\right) \tag{5.171}$$

A r-form is simple if A_r is a blade ($A_r = a_1 \wedge \dots \wedge a_r$).

5.6.3 Dual of a Vector Space

Let V be a n-dimensional vector space. The set of all linear maps $f : V \to \mathcal{R}$ is denoted V^* and is a vector space called the dual space of V. V^* is a n-dimensional vector space since for $f, g : V \to \mathcal{R}$, $x \in V$, $\alpha \in \mathcal{R}$ and we define

$$\begin{aligned} (f+g)(x) &\equiv f(x) + g(x) && (5.172)\\ (\alpha f)(x) &\equiv \alpha f(x) && (5.173)\\ 0(x) &\equiv 0 && (5.174) \end{aligned}$$

Then by the linearity of $f(x)$ and $g(x)$, $(f+g)(x)$, $\alpha f(x)$, and $0(x)$ are also linear functions of x and V^* is a vector space.

Let $\mathbf{e}_i$ be a basis for V and define $\sigma^i \in V^*$ by

$$\sigma^i\left(\mathbf{e}_j\right) = \delta^i_j \tag{5.175}$$

so that if $x = x^i\mathbf{e}_i \in V$ then

$$\sigma^i(x) = \sigma^i\left(x^j\mathbf{e}_j\right) = x^i \tag{5.176}$$

so that for any $f \in V^*$

$$f(x) = f\left(x^i\mathbf{e}_i\right) = x^i f(\mathbf{e}_i) \tag{5.177}$$

Now assume that

$$\begin{aligned} 0 &= a_i\sigma^i(\mathbf{e}_j) && (5.178)\\ &= a_i\delta^i_j && (5.179)\\ &= a_j && (5.180) \end{aligned}$$

so the σ^i are linearly independent. Now assume $f \in V^*$. Then

$$\begin{aligned} f(x) &= f\left(x^i\mathbf{e}_i\right) && (5.181)\\ &= x^i f(\mathbf{e}_i) && (5.182)\\ &= f(\mathbf{e}_i)\,\sigma^i(x) && (5.183)\\ &= \left(f(\mathbf{e}_i)\,\sigma^i\right)(x) && (5.184) \end{aligned}$$

Thus

$$f = f(\mathbf{e}_i)\,\sigma^i \tag{5.185}$$

and the σ^i form a basis for V^*. If the $\mathbf{e}_i$ form an orthonormal basis for V (remember that orthonomal implies orthogonal and orthogonal implies that a dot product is defined since the dot product is used to define orthogonal) then

$$\sigma^i(x) = \mathbf{e}_i \cdot x. \tag{5.186}$$

Since for 1-forms, $\alpha : V \to \mathcal{R}$, we have $\alpha \in V^*$. The most general 1-form can be written

$$\alpha(v) = \alpha_i\sigma^i(v). \tag{5.187}$$

If α is a 2-form, $\alpha(v_1, v_2)$, the bases are $\sigma^i\hat{\wedge}\sigma^j$, and the most general 2-form is written as

$$\alpha(v_1, v_2) = \sum_{i<j} \alpha_{ij}\sigma^i(v_1)\,\hat{\wedge}\sigma^j(v_2) \tag{5.188}$$

since $\sigma^i\hat{\wedge}\sigma^i = 0$ and $\sigma^i\hat{\wedge}\sigma^j = -\sigma^i\hat{\wedge}\sigma^j$ from the definition in equation 5.167.

5.6.4 Standard Definition of a Manifold

Let $\mathcal{M}^n$ be any set[2] that has a covering of subsets, $\mathcal{M}^n = U \cup V \cup \ldots$ such that

1. For each subset U there is a one to one mapping $\phi_U : U \rightarrow \mathcal{R}^n$ where $\phi_U(U)$ is an open subset of $\mathcal{R}^n$.

2. Each $\phi_U(U \cap V)$ is an open subset of $\mathcal{R}^n$.

3. The overlap maps
$$f_{UV} = \phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \rightarrow \mathcal{R}^n \tag{5.189}$$
or equivalently the compound maps
$$\phi_U(U \cap V) \xrightarrow{\phi_U^{-1}} \mathcal{M}^n \xrightarrow{\phi_V} \mathcal{R}^n \tag{5.190}$$
are differentiable.

4. Take a maximal atlas of coordinate patches $\{(U, \phi_U), (V, \phi_V), \ldots\}$ and define a topology for $\mathcal{M}^n$ by defining that a subset $W \subset \mathcal{M}^n$ is open if for any $p \in W$ there is a (U, ϕ_U) such that $p \in U \subset W$.

If the resulting topology for $\mathcal{M}^n$ is Hausdorff and has a countable base we say $\mathcal{M}^n$ is an n-dimensional differentiable manifold (look it up since I don't know what it means[3]).

Figure 5.3 (page 94) shows the relationships between the Manifold and the coordinate patch mappings.

If $f : \mathcal{M}^n \rightarrow \mathcal{R}$ is a real valued function on the manifold $\mathcal{M}^n$ it is differentiable if $f_U = f \circ \phi_U^{-1}$ is differentiable with respect to the coordinates $\{x_U^1, \ldots, x_U^n\}$ for any coordinate patch (U, ϕ_U). The real scalars $\{x_U^1, \ldots, x_U^n\}$ (denoted by the tuple $x = (x_U^1, \ldots, x_U^n)$) form a coordinate system for $\phi_U(U) \subset \mathcal{R}^n$. In the future we shall simply say that f is differentiable if f_U is differentiable. Likewise we will shall usually omit the process of replacing f by its composition $f \circ \phi_U^{-1}$, thinking of f as directly expressible as a function $f(x) = f(x_U^1, \ldots, x_U^n)$ of any local coordinates.

Now let $p \in U \cap V \subset \mathcal{M}^n$ and
$$x_U = \left(x_U^1, \ldots, x_U^n\right) = \phi_U(p) \text{ and } x_V = \left(x_V^1, \ldots, x_V^n\right) = \phi_V(p)\,.$$

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[2] This section is based upon sections 1.2 and 2.1 in "The Geometry of Physics, An Introduction (Second Edition)," by T. Frankel

[3] G. Simmons, *Topology and Modern Analysis*, McGraw-Hill,1963

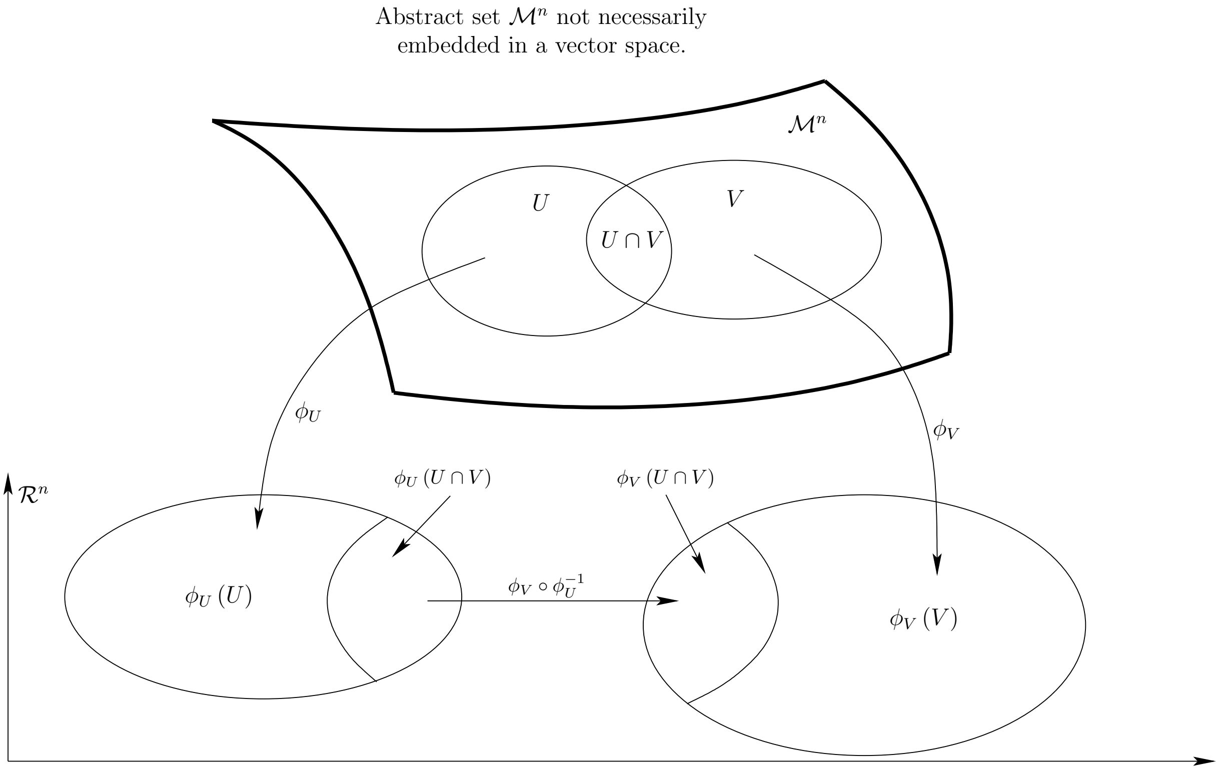

Figure 5.3: Fundamental Manifold Mappings.

Then

$$x_U = \phi_U \circ \phi_V^{-1}\left(x_V\right) = x_U\left(x_V\right) \text{ and } x_V = \phi_V \circ \phi_U^{-1}\left(x_U\right) = x_V\left(x_U\right).$$

Let $X_U = \left(X_U^1, \ldots, X_U^n\right)$ and consider the linear approximation to $\phi_V \circ \phi_U^{-1}\left(x_U + hX_U\right)$ where h is a small number. Then in the linear approximation

$$\phi_V \circ \phi_U^{-1}\left(x_U + hX_U\right) = x_V\left(x_U\right) + h\left[\frac{\partial x_V^i}{\partial x_U^j}\right] X_U^T = x_V + hX_V \tag{5.191}$$

so that

$$X_V^i = \frac{\partial x_V^i}{\partial x_U^j} X_U^j \tag{5.192}$$

where $\left[\dfrac{\partial x_V^i}{\partial x_U^j}\right]$ is the Jacobian matrix of the coordinate transformation from one coordinate patch to another and X_U^T is the transpose (colume vector) of the tuple X_U (row vector).

5.6.5 Tangent Space

Now consider the case of a vector manifold in Euclidian space. The definition of the tangent vector generalizes the concept of the directional derivative in $\mathcal{R}^n$. if X_p is a vector at point $p \in \mathcal{R}^n$ and $f : \mathcal{R}^n \to \mathcal{R}$ is a C^∞ function in the neighborhood of p then define

$$X_p\left(f\right) \equiv X_p \cdot \left.\nabla f\right|_p. \tag{5.193}$$

If f and g are C^∞ and $\alpha, \beta \in \mathcal{R}$ we have (from the properties of $\cdot$ and ∇)

$$X_p\left(\alpha f + \beta g\right) = \alpha X_p\left(f\right) + \beta X_p\left(g\right) \tag{5.194}$$

$$X_p\left(fg\right) = f\left(p\right) X_p\left(g\right) + g\left(p\right) X_p\left(f\right) \tag{5.195}$$

Let $C^\infty\left(p\right)$ denote the set of real functions that are C^∞ on some neighborhood of $p \in \mathcal{M}^n$. The generalization of the tangent vector to an abstract manifold at a point $p \in \mathcal{M}^n$ is defined as a real valued function $X_p : C^\infty\left(p\right) \to \mathcal{R}$ such that ($f, g \in C^\infty\left(p\right)$ and $\alpha \in \mathcal{R}$)

$$X_p\left(f + g\right) = X_p\left(f\right) + X_p\left(g\right) \tag{5.196}$$

$$X_p\left(\alpha f\right) = \alpha X_p\left(f\right) \tag{5.197}$$

$$X_p\left(fg\right) = f\left(p\right) X_p\left(g\right) + g\left(p\right) X_p\left(f\right). \tag{5.198}$$

A tangent vector is often called a derivation on $C^{\infty}(p)$. For a given coordinate patch (U,ϕ_U) we define the tangent space basis vectors $\left(\dfrac{\partial}{\partial x_U^i}\right)_p$ by

$$\left(\frac{\partial}{\partial x_U^i}\right)_p f \equiv \left.\frac{\partial\left(f\circ\phi_U^{-1}\right)}{\partial x_U^i}\right|_{\phi_U(p)}. \tag{5.199}$$

Since they are partial derivatives the $\left(\dfrac{\partial}{\partial x_U^i}\right)_p$ trivially satisfy equations 5.196, 5.197, and 5.198. The general tangent vector is then given by the scalar linear differential operator

$$X_p = X_U^i\left(\frac{\partial}{\partial x_U^i}\right)_p \tag{5.200}$$

Thus we require for X_p

$$X_p(f) = X_U^i\left(\frac{\partial}{\partial x_U^i}\right)_p f = X_V^j\left(\frac{\partial}{\partial x_V^j}\right)_p f \tag{5.201}$$

for all $p\in U\cap V$. Thus

$$X_U^i\left(\frac{\partial}{\partial x_U^i}\right)_p f = X_V^j\left(\frac{\partial}{\partial x_V^j}\right)_p f \tag{5.202}$$

$$X_U^i\left(\frac{\partial x_V^k}{\partial x_U^i}\right)_p\left(\frac{\partial}{\partial x_V^k}\right)_p f = X_V^j\left(\frac{\partial}{\partial x_V^j}\right)_p f \tag{5.203}$$

and

$$X_V^j = X_U^i\left(\frac{\partial x_V^j}{\partial x_U^i}\right)_p \tag{5.204}$$

is the transformation law for the components, X_U^i, of the tangent vectors.

We have established an isomorphism between the tuple $(X_U^1,\dots,X_U^n)$ that transforms as in equation 5.192 and the scalar linear operator X_p so that

$$\left(X_U^1,\dots,X_U^n\right)\Leftrightarrow X_U^i\frac{\partial}{\partial x_u^i} \tag{5.205}$$

since

$$X_U^i\frac{\partial f}{\partial x_U^i} = X_V^j\frac{\partial f}{\partial x_V^i},\ \forall\, f:\mathcal{M}^n\to\mathcal{R} \tag{5.206}$$

implies equation 5.192. A vector field on a manifold is then a tangent vector with coefficients, $X^i(p)$ that are functions of the position p on the manifold so that $(x_U = (x_U^1, \ldots, x_U^n))$

$$X(p) = X_U(x_U) = X_U^i(x_U)\left(\frac{\partial}{\partial x_U^i}\right)_p = X_U^i\left(x_U^1, \ldots, x_U^n\right)\left(\frac{\partial}{\partial x_U^i}\right)_p \tag{5.207}$$

and the coefficients $X_U^i(x_U)$ transform under a change of basis according to equation 5.204.

5.6.6 Differential Forms and the Dual Space

If $f : \mathcal{M}^n \rightarrow \mathcal{R}$ we define the differential of f at $p \in \mathcal{M}^n$, $df : \mathcal{M}_p^n \rightarrow \mathcal{R}$ where $X_p \in \mathcal{M}_p^n$ as the linear functional

$$df(X) \equiv X_p(f) \tag{5.208}$$

so that in general case of X being a vector field (we are supressing the the patch index U)

$$df(X(p)) = df\left(X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p\right) = X^i(p)\left.\frac{\partial f}{\partial x^i}\right|_p. \tag{5.209}$$

If $f(p) = x^i(p)$ are the coordinate functions we have

$$dx^i\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = \left.\frac{\partial x^i}{\partial x^j}\right|_p = \delta_j^i \tag{5.210}$$

and

$$dx^i(X(p)) = dx^i\left(X^j(p)\left(\frac{\partial}{\partial x^j}\right)_p\right) = X^j(p)\,dx^i\left(\left(\frac{\partial}{\partial x^j}\right)_p\right) = X^i(p). \tag{5.211}$$

Consider what this means in a vector manifold where the point p on the manifold is a vector in the embedding vector space. Then the coordinate functions $x^i(p)$ can be inverted and the manifold defined by $p(x^1, \ldots, x^n)$. Knowing the tuple $(x^1, \ldots, x^n)$ lets one calculate the vector p on the manifold. Then the basis tangent vectors are $e_i = \dfrac{\partial p}{\partial x^i}$ and the general tangent vector is $X_p = X^i e_i = X^i \dfrac{\partial p}{\partial x^i}$. Now let $f(p) = f(x^1, \ldots, x^n) : \mathcal{R}^n \rightarrow \mathcal{R}$ be a function from the manifold to the scalars and denote the directional derivative of f as

$$df(X_p) = X \cdot \nabla f|_p = X^i \left.\frac{\partial f}{\partial x^i}\right|_p \tag{5.212}$$

which is the same as equation 5.207 and the justification for defining the tangent vectors as the linear scalar differential operator $X_p = X^i \left.\dfrac{\partial}{\partial x^i}\right|_p$. What is called the differential operator in the language of differential forms becomes the directional derivative on a vector manifold.

The dx^i form a basis for the dual space $\mathcal{M}_p^{n*}$. The most general element of $\mathcal{M}_p^{n*}$ can then be written as

$$\alpha = \alpha_i dx^i. \tag{5.213}$$

The linear functional $\alpha \in \mathcal{M}_p^{n*}$ is called a covariant vector, covector, or 1-form. If α_i is a function of p it is called a covector field.

α is an r-form if

$$\alpha = \alpha_{i_1 \dots i_r}(p)\, dx^{i_1} \hat{\wedge} \dots \hat{\wedge} dx^{i_r} \tag{5.214}$$

where $i_1 < i_2 < \cdots < i_r$. The exterior derivative of α is an $r+1$ form defined by

$$d\alpha \equiv \left(d\alpha_{i_1 \dots i_r}(p)\right) \hat{\wedge} dx^{i_1} \hat{\wedge} \dots \hat{\wedge} dx^{i_r}. \tag{5.215}$$

For example if $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3$ then the exterior derivative of the 1-form α is

$$d\alpha = \left(\frac{\partial \alpha_2}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^2}\right) dx^1 \hat{\wedge} dx^2 + \left(\frac{\partial \alpha_3}{\partial x^1} - \frac{\partial \alpha_1}{\partial x^3}\right) dx^1 \hat{\wedge} dx^3 + \left(\frac{\partial \alpha_3}{\partial x^2} - \frac{\partial \alpha_2}{\partial x^3}\right) dx^2 \hat{\wedge} dx^3 \tag{5.216}$$

and if $\alpha = \alpha_{12} dx^1 \hat{\wedge} dx^2 + \alpha_{13} dx^1 \hat{\wedge} dx^3 + \alpha_{23} dx^2 \hat{\wedge} dx^3$ then the exterior derivative of the 2-form α is

$$d\alpha = \left(\frac{\partial \alpha_{12}}{\partial x^1} - \frac{\partial \alpha_{13}}{\partial x^2} + \frac{\partial \alpha_{23}}{\partial x^3}\right) dx^1 \hat{\wedge} dx^2 \hat{\wedge} dx^3. \tag{5.217}$$

5.6.7 Connecting Differential Forms to Geometric Calculus

To connect differential forms to the geometric calculus we must establish a correspondence between the fundamental theorem of geometric calculus and the generalize Stoke's theorem of differential forms. For simplicity, since we do not wish to use the most general formulation of Stoke's theorem we will use the treatment in "Calculus on Manifolds" by Spivak (`http://faculty.ksu.edu.sa/fawaz/482/Books/Spivak_Calculus%20on%20manifolds.pdf`). In this treatment a manifold is embedded in the Euclidian space $\Re^n$ so that the manifold and its boundry in Spivak are submanifolds of $\Re^n$ as are also the manifolds V and ∂V which are the domains of integration in the fundamental theorem of geometric calculus (eq 4.81). The fundamental theorem of geometric calculus is more general than the generalized Stoke's theorem since the linear functional in the integrals can be multivector valued fields. The appropriate form of geometric calculus theorem is (for explicitness subscripts in parenthesis indicate the grade of a multivector field or the dimension of a manifold or it's boundary, of the rank of a differential form)

$$\oint_{\partial V_{(r)}} \Omega_{(r)} \cdot dS_{(r)} = (-1)^r \int_{V_{(r+1)}} \left(\nabla \wedge \Omega_{(r)}\right) \cdot dV_{(r+1)}. \tag{5.218}$$

Equation 5.218 corresponds to the generalized Stokes theorem in differential forms

$$\oint_{\partial V_{(r)}} \boldsymbol{\omega}_{(r)} = \int_{V_{(r+1)}} d\boldsymbol{\omega}_{(r+1)}. \tag{5.219}$$

The differential forms $\boldsymbol{\omega}_{(r)}$ and $d\boldsymbol{\omega}_{(r+1)}$ are alternating tensors (antisymmetric) of ranks r and $r+1$ respectively. We now must expand equations (5.218) and (5.219) in terms of their components to show equivalence.

Chapter 6

Multivector Calculus

In order to develop multivector Lagrangian and Hamiltonian methods we need to be able to take the derivative of a multivector function with respect to another multivector. One example of this are Lagrangians that are function of spinors (which are even multivectors) as in quantum field theory. This chapter contains a brief description of the mechanics of multivector derivataives.

6.1 New Multivector Operations

Define the index $i_{\{r\}} = (i_1, i_2, \dots, i_r)$ where $r \le n$ the dimension of the vector space and $P\left(i_{\{r\}}\right)$ is the union of 0 and the set of ordered tuples $i_{\{r\}} = (i_1, i_2, \dots, i_r)$ defined by

$$P\left(i_{\{r\}}\right) \equiv \left\{ \begin{array}{c} \{0 \text{ if } r = 0\} \cup \\ \{(i_1, i_2, \dots, i_r) \text{ such that } i_1 < i_2 < \cdots < i_r \text{ and } 1 \le i_j \le n \text{ for } 1 \le j \le r\} \end{array} \right\}. \tag{6.1}$$

Essentially $P\left(i_{\{r\}}\right)$ is an index set that enumerates the r-grade bases of the geometric algebra of an n-dimensional vector space where $0 \le r \le n$. Then define the basis blades

$$e_{i_{\{r\}}} \equiv e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} \tag{6.2}$$

where $e_{i_{\{0\}}} = e_0 = 1$ and

$$e^{i_{\{r\}}} \equiv e^{i_r} \wedge e^{i_{r-1}} \wedge \ldots \wedge e^{i_1} \tag{6.3}$$

where $e^{i_{\{0\}}} = e^0 = 1$ and the e_i's form a basis for the multivector space. With these definitions

$$e^{i_{\{r\}}} \cdot e_{j_{\{r\}}} = \delta^{i_{\{r\}}}_{j_{\{r\}}}. \tag{6.4}$$

The multivector X can now be written as

$$X = \sum_{r=0}^{n} \sum_{i_{\{r\}} \in P\left(i_{\{r\}}\right)} X^{i_{\{r\}}} e_{i_{\{r\}}} = \sum_{r=0}^{n} \sum_{i_{\{r\}} \in P\left(i_{\{r\}}\right)} X_{i_{\{r\}}} e^{i_{\{r\}}}. \tag{6.5}$$

We now generalize the Einstein summation convention so we can write $X = X^{i_{\{r\}}} e_{i_{\{r\}}}$. Now it is clear that X is a 2^n dimensional vector with components $X^{i_{\{r\}}}$. For example for $n = 3$

$$\begin{aligned} X &= \langle X \rangle_0 + \langle X \rangle_1 + \langle X \rangle_2 + \langle X \rangle_3 && (6.6) \\ &= X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + \\ &\quad X^{12} e_1 \wedge e_2 + X^{13} e_1 \wedge e_3 + X^{23} e_2 \wedge e_3 + X^{123} e_1 \wedge e_2 \wedge e_3. && (6.7) \end{aligned}$$

From the properties of the dot product of blades developed in section 5.6.1 we have for $r > 1$

$$\begin{aligned} e_{i_{\{r\}}} \cdot e_{j_{\{r\}}} &= (e_{i_1} \wedge \ldots \wedge e_{i_r}) \cdot (e_{j_1} \wedge \ldots \wedge e_{j_r}) && (6.8) \\ &= \delta_{i_{\{r\}} j_{\{r\}}} e_{i_{\{r\}}}^2 && (6.9) \end{aligned}$$

Now define the scalar product of two multivectors A and B by

$$A * B \equiv \langle AB \rangle . \tag{6.10}$$

Then

$$\begin{aligned} &\langle A \rangle_r * \langle B \rangle_s = 0 \text{ if } r \neq s && (6.11) \\ &\langle A \rangle_r * \langle B \rangle_r = \langle A \rangle_r \cdot \langle B \rangle_r = \langle B \rangle_r * \langle A \rangle_r \ \text{ if } r \neq 0 && (6.12) \\ &\langle A \rangle_0 * \langle B \rangle_0 = \langle A \rangle_0 \langle B \rangle_0 = \langle A \rangle \langle B \rangle && (6.13) \\ &A * B = \sum_{r=0}^{n} \langle A \rangle_r * B = \sum_{r=0}^{n} \langle A \rangle_r * \langle B \rangle_r = \langle A \rangle \langle B \rangle + \sum_{r=1}^{n} \langle A \rangle_r \cdot \langle B \rangle_r && (6.14) \end{aligned}$$

by grade counting arguments and the orthogonality properties of basis blades of grade greater that 1. Note that since $\langle A \rangle_r$ is a sum of blades each defining a different subspace we have by equation 5.145 that the blades that compose $\langle A \rangle_r$ are orthogonal to one another under the $*$ and $\cdot$ operations. Also

$$\begin{aligned} A * B &= \langle AB \rangle = \langle BA \rangle = B * A && (6.15) \\ A * (\alpha B + \beta C) &= \alpha A * B + \beta A * C && (6.16) \\ A * B &= A^\dagger * B^\dagger. && (6.17) \end{aligned}$$

We also have

$$e^{i_{\{r\}}}\cdot e_{j_{\{r\}}}=e^{i_{\{r\}}}*e_{j_{\{r\}}}=\delta^{i_{\{r\}}}_{j_{\{r\}}}\text{ for }r>0. \tag{6.18}$$

We now use the scalar product to define the scalar magnitude $|A|$ of a multivector A by

$$|A|^2\equiv A^\dagger *A=\sum_{r=0}^{n}\left|\langle A\rangle_r\right|^2 \tag{6.19}$$

Now define a super metric tensor for the entire geometric algebra vector space by

$$G_{i_{\{r\}}j_{\{s\}}}\equiv e_{i_{\{r\}}}*e_{j_{\{s\}}}=\delta_{r,s}\left\{\begin{array}{c}1\text{ for }r=0\\ g_{i_{\{1\}}j_{\{1\}}}\text{ for }r=1\\ \delta_{i_{\{r\}}j_{\{r\}}}\left(e_{i_{\{r\}}}\right)^2\text{ for }r>1\end{array}\right\} \tag{6.20}$$

and

$$G^{i_{\{r\}}j_{\{s\}}}\equiv e^{i_{\{r\}}}*e^{j_{\{s\}}}=\delta^{r,s}\left\{\begin{array}{c}1\text{ for }r=0\\ g^{i_{\{1\}}j_{\{1\}}}\text{ for }r=1\\ \delta^{i_{\{r\}}j_{\{r\}}}\left(e_{i_{\{r\}}}\right)^{-2}\text{ for }r>1\end{array}\right\} \tag{6.21}$$

The super metric tensors have only diagonal entries except for $G_{i_{\{1\}}j_{\{1\}}}$ and $G^{i_{\{1\}}j_{\{1\}}}$. Due to the block nature of the G tensors ($G_{i_{\{r\}}j_{\{s\}}}=G^{i_{\{r\}}j_{\{s\}}}=0$ for $r\neq s$) we can write

$$e_{j_{\{r\}}}=e^{i_{\{r\}}}\left(e_{i_{\{r\}}}*e_{j_{\{r\}}}\right)=e^{i_{\{r\}}}\left(e_{j_{\{r\}}}*e_{i_{\{r\}}}\right) \tag{6.22}$$

$$e^{j_{\{r\}}}=e_{i_{\{r\}}}\left(e^{i_{\{r\}}}*e^{j_{\{r\}}}\right)=e_{i_{\{r\}}}\left(e^{j_{\{r\}}}*e^{i_{\{r\}}}\right). \tag{6.23}$$

An additional relation that we need to prove is (no summation in this case)

$$e^{i_{\{r\}}}e_{i_{\{r\}}}=e^{i_{\{r\}}}\cdot e_{i_{\{r\}}}=1. \tag{6.24}$$

First consider the case for $r=1$

$$\begin{aligned}e^{i_{\{1\}}}e_{i_{\{1\}}}&=e^{i_1}e_{i_1}=e^{i_1}\cdot e_{i_1}+e^{i_1}\wedge e_{i_1}\\ &=1+g^{j_1i_1}e_{j_1}\wedge e_{i_1}=1+\sum_{j_1<i_1}g^{j_1i_1}\left(e_{j_1}\wedge e_{i_1}+e_{i_1}\wedge e_{j_1}\right)=1.\end{aligned} \tag{6.25}$$

Now consider the case $r>1$. Since $e_{i_{\{r\}}}$ and $e^{i_{\{r\}}}$ are blades that define the same r-dimesional subspaces they can be written as the geometric product of the same r orthogonal vectors $o_1,\dots,o_r$ to within a scale factor so that

$$\begin{aligned}e^{i_{\{r\}}}e_{i_{\{r\}}}&=\left(e^{i_r}\wedge\dots\wedge e^{i_1}\right)\left(e_{i_1}\wedge\dots\wedge e_{i_r}\right)\\ &=\alpha\beta\left(o_r\dots o_1\right)\left(o_1\dots o_r\right)=\alpha\beta o_1^2\dots o_r^2=e^{i_{\{r\}}}\cdot e_{i_{\{r\}}}=1\end{aligned} \tag{6.26}$$

since by equation 6.26 $e^{i_{\{r\}}}e_{i_{\{r\}}}=\left\langle e^{i_{\{r\}}}e_{i_{\{r\}}}\right\rangle$ is a scalar.

6.2 Derivatives With Respect to Multivectors

We start by discussing exactly what we mean we say $F(X)$ is a funtion of a multivector X. First the domain and range of $F(X)$ are both multivectors in a 2^n-dimensional vector space formed from the multivector space of the n-dimensional base vector space. Another way of stating this is that the geometric algebra of the base vector space is generated by the n-grade pseudoscalar, I_n, of the base vector space . The domain of F is the multivector space of the geometric algebra defined by the normalized pseudoscalar I_n where $I_n^2 = \pm 1$. Thus we can consider X to be an element of the 2^n dimensional vector space defined by I_n. The partial dervatives of $F(X)$ are then the multivectors $\dfrac{\partial F}{\partial X^{i_{\{r\}}}}$ and there are 2^n of them. The definition of the multivector directional derivative ∂_X is

$$(A*\partial_X)\,F \equiv \lim_{h\rightarrow 0}\frac{F(X+hA)-F(X)}{h}. \tag{6.27}$$

Thus

$$\begin{aligned}(A*\partial_X)\,F &= \lim_{h\rightarrow 0}\frac{F(X)+hA^{i_{\{r\}}}\dfrac{\partial F}{\partial X^{i_{\{r\}}}}-F(X)}{h} && (6.28)\\ &= A^{i_{\{r\}}}\frac{\partial F}{\partial X^{i_{\{r\}}}} && (6.29)\\ &= A^{i_{\{r\}}}e_{i_{\{r\}}}*e^{j_{\{r\}}}\frac{\partial F}{\partial X^{j_{\{r\}}}} && (6.30)\\ &= A*e^{j_{\{r\}}}\frac{\partial F}{\partial X^{j_{\{r\}}}} && (6.31)\end{aligned}$$

so that in terms of components

$$\partial_X = e^{j_{\{r\}}}\frac{\partial}{\partial X^{j_{\{r\}}}}. \tag{6.32}$$

Note that we have put parenthesis around $(A*\partial_X)$ to remind ourselves (Doran and Lasenby do not do this) that $A*\partial_X$ is a scalar operater in exactly the same way that $a\cdot\nabla$, $a\cdot\partial$, and $a\cdot D$ are scalar operators. While not explicitly stated in D&L or Hestenes, $*$ must have the same precendence as $\cdot$ and higher than any of the other product of geometric algebra. The multivector derivative ∂_X is calculated by letting A take on the values of the bases for the geometric algebra.

Another notation used for the multivector derivative is

$$\underline{F}_X(A) = \underline{F}(A) \equiv (A*\partial_X)\,F(X). \tag{6.33}$$

The form $\underline{F}(A)$ would be used if it is implicitely understood that the multivector derivative is to be taken with respect to X.

We can now define the adjoint of $\underline{F}(A)$ by

$$\overline{F}(B) \equiv \partial_A \left\langle \underline{F}(A)\, B \right\rangle \tag{6.34}$$

This make sense when we consider

$$\begin{aligned}
\left\langle A\overline{F}(B) \right\rangle &= \left\langle A\partial_C \left\langle \underline{F}(C)\, B \right\rangle \right\rangle \\
&= \left\langle A\partial_C \right\rangle \left\langle \underline{F}(C)\, B \right\rangle \\
&= (A * \partial_C) \left\langle \underline{F}(C)\, B \right\rangle \\
&= \lim_{h\to 0} \frac{\left\langle \underline{F}(C + hA)\, B - \underline{F}(C)\, B \right\rangle}{h} \\
&= \left\langle \underline{F}(A)\, B \right\rangle \\
A * \overline{F}(B) &= \underline{F}(A) * B.
\end{aligned} \tag{6.35}$$

When A and B are vectors a and b and $\underline{F}(a)$ is a vector-valued linear function of a we have

$$\begin{aligned}
a * \overline{F}(b) &= \underline{F}(a) * b \\
a \cdot \overline{F}(b) &= \underline{F}(a) \cdot b
\end{aligned} \tag{6.36}$$

which recovers the original definition of the adjoint.

Note: Consider equation 6.35 for the case that A and B are pure grade, but not the same grade. For example let B be a bivector, $\overline{F}(B)$ a vector, and A a vector. Then $A * \overline{F}(B) = A \cdot \underline{F}(B)$ is a scalar, but then $\underline{F}(A)$ must be a bivector for $\underline{F}(A) * B$ to be non-zero. In general if A and $\overline{F}(B)$ are pure grade they must be the same grade then $\underline{F}(A)$ is the same pure grade as B and we can write $A \cdot \overline{F}(B) = B \cdot \underline{F}(A)$.

Product rule -

$$\begin{aligned}
(A*\partial_X)(FG) &= \lim_{h\to 0}\frac{F(X+hA)\,G(X+hA)-F(X)\,G(X)}{h}\\
&= \lim_{h\to 0}\frac{\left(F(X)+hA^{i_{\{r\}}}\dfrac{\partial F}{\partial X^{i_{\{r\}}}}\right)\left(G(X)+hA^{j_{\{r\}}}\dfrac{\partial G}{\partial X^{j_{\{r\}}}}\right)-F(X)\,G(X)}{h}\\
&= A^{i_{\{r\}}}\frac{\partial F}{\partial X^{i_{\{r\}}}}G+A^{j_{\{r\}}}F\frac{\partial G}{\partial X^{j_{\{r\}}}}\\
&= A^{i_{\{r\}}}\left(\frac{\partial F}{\partial X^{i_{\{r\}}}}G+F\frac{\partial G}{\partial X^{i_{\{r\}}}}\right)\\
&= A^{i_{\{r\}}}e_{i_{\{r\}}}*e^{j_{\{r\}}}\frac{\dot{\partial}}{\partial X^{j_{\{r\}}}}\left(\dot{F}G+F\dot{G}\right)\\
&= \left(A*\dot{\partial}_X\right)\left(\dot{F}G+F\dot{G}\right) \qquad (6.37)
\end{aligned}$$

so that the product rule is

$$\partial_X(FG)=\dot{\partial}_X\left(\dot{F}G+F\dot{G}\right). \qquad (6.38)$$

Chain rule -

$$\begin{aligned}
(A*\partial_X)F(G(X)) &= \lim_{h\to 0}\frac{F(G(X+hA))-F(G(X))}{h}\\
&= \lim_{h\to 0}\frac{F\left(G(X)+hA^{i_{\{r\}}}\dfrac{\partial G}{\partial X^{i_{\{r\}}}}\right)-F(G(X))}{h}\\
&= \lim_{h\to 0}\frac{F(G(X))+hA^{i_{\{r\}}}\dfrac{\partial G^{j_{\{r\}}}}{\partial X^{i_{\{r\}}}}\dfrac{\partial F}{\partial G^{j_{\{r\}}}}-F(G(X))}{h}\\
&= A^{i_{\{r\}}}\frac{\partial G^{j_{\{r\}}}}{\partial X^{i_{\{r\}}}}\frac{\partial F}{\partial G^{j_{\{r\}}}}\\
&= A^{i_{\{r\}}}\frac{\partial G^{j_{\{r\}}}}{\partial X^{i_{\{r\}}}}e_{j_{\{r\}}}*e^{k_{\{r\}}}\frac{\partial F}{\partial G^{k_{\{r\}}}}\\
&= \left(A^{i_{\{r\}}}\frac{\partial G}{\partial X^{i_{\{r\}}}}\right)*\partial_G F\\
&= ((A*\partial_X)\,G)*\partial_G F \qquad (6.39)
\end{aligned}$$

If $g(X)$ is a scalar function of a multivector X and $f(g)$ is a scalar function of a scalar then

$$((A*\partial_X)\,g)\frac{df}{dg}=\left(A*\dot{\partial}_X\right)\dot{g}\frac{df}{dg} \qquad (6.40)$$

and

$$\partial_X f\left(g\left(X\right)\right) = \left(\partial_X g\right)\frac{df}{dg}. \tag{6.41}$$

One other general relationship of use is the evaluation of $\partial_A\left(\left(A*\partial_X\right)F\right)$

$$\begin{aligned}\left(B*\partial_A\right)\left(\left(A*\partial_X\right)F\right) &= \lim_{h\rightarrow 0}\frac{\left(\left(\left(A+hB\right)*\partial_X\right)F\right)-\left(A*\partial_X\right)F}{h}.\\ &= \left(B*\partial_X\right)F \qquad (6.42)\end{aligned}$$

so that

$$\partial_A\left(\left(A*\partial_X\right)F\right) = \partial_X F. \tag{6.43}$$

For simple derivatives we have the following -

1. $\partial_X\left(A*X\right)$:

$$\begin{aligned}\left(B*\partial_X\right)\left(A*X\right) =& \lim_{h\rightarrow 0}\frac{\left\langle A\left(X+hB\right)-AX\right\rangle}{h}\\ =& \left\langle AB\right\rangle = B*A\\ \partial_X\left(A*X\right) =& A \qquad (6.44)\end{aligned}$$

2. $\partial_X\left(X*X^\dagger\right)$:

$$\begin{aligned}\left(A*\partial_X\right)\left(X*X^\dagger\right) =& \lim_{h\rightarrow 0}\frac{\left\langle\left(X+hA\right)\left(X+hA\right)^\dagger - XX^\dagger\right\rangle}{h}\\ =& \left\langle XA^\dagger + AX^\dagger\right\rangle = X*A^\dagger + A*X^\dagger\\ =& 2A*X^\dagger \qquad (6.45)\\ \partial_X\left(X*X^\dagger\right) =& 2X^\dagger \qquad (6.46)\\ \partial_{X^\dagger}\left(X*X^\dagger\right) =& 2X \qquad (6.47)\end{aligned}$$

3. $\partial_X\left(X*X\right)$:

$$\begin{aligned}\left(A*\partial_X\right)\left(X*X\right) =& \lim_{h\rightarrow 0}\frac{\left\langle\left(X+hA\right)\left(X+hA\right) - XX\right\rangle}{h}\\ =& \left\langle XA + AX\right\rangle = X*A + A*X\\ =& 2A*X\\ \partial_X\left(X*X\right) =& 2X \qquad (6.48)\\ \partial_{X^\dagger}\left(X^\dagger*X^\dagger\right) =& 2X^\dagger \qquad (6.49)\end{aligned}$$

4. $\partial_X \left(X^\dagger * X^\dagger\right)$:

$$\begin{aligned}
(A * \partial_X) \left(X^\dagger * X^\dagger\right) =& \lim_{h \to 0} \frac{\left\langle (X + hA)^\dagger (X + hA)^\dagger - X^\dagger X^\dagger \right\rangle}{h} \\
=& \left\langle X^\dagger A^\dagger + A^\dagger X^\dagger \right\rangle = X^\dagger * A^\dagger + A^\dagger * X^\dagger \\
=& 2A * X \\
\partial_X \left(X^\dagger * X^\dagger\right) =& 2X \qquad (6.50) \\
\partial_{X^\dagger} (X * X) =& 2X^\dagger \qquad (6.51)
\end{aligned}$$

5. $\partial_X \left(|X|^k\right)$:

$$\begin{aligned}
\partial_X \left(|X|^k\right) &= \partial_X \left(\left(|X|^2\right)^{\frac{k}{2}}\right) \\
&= 2\frac{k}{2} \left(|X|^2\right)^{\frac{k}{2}-1} X^\dagger \\
&= k\, |X|^{k-2}\, X^\dagger \qquad (6.52)
\end{aligned}$$

by the chain rule.

6.3 Calculus for Linear Functions

Let $\mathbf{f}(a)$ be a linear function mapping the vector space onto itself and the e_i the basis for the vector space then

$$\begin{aligned}
\mathbf{f}(a) &= \mathbf{f}\left(a^j e_j\right) \\
&= \mathbf{f}(e_j)\, a^j \\
&= e^i \left(e_i \cdot \mathbf{f}(e_j)\right) a^j \qquad (6.53)
\end{aligned}$$

The coefficient matrix for a linear function $\mathbf{f}(a)$ with respect to basis e_i is defined as

$$f_{ij} \equiv e_i \cdot \mathbf{f}(e_j) \qquad (6.54)$$

and

$$\mathbf{f}(a) = e^i f_{ij} a^j \qquad (6.55)$$

Now consider the derivatives of the scalar $\mathbf{f}(b)\cdot c$ with respect to the f_{ij}

$$\begin{aligned}\partial_{f_{ij}}\mathbf{f}(b)\cdot c &= \partial_{f_{ij}}\left(f_{lk}b^kc^l\right)\\ &= \delta_{il}\delta_{jk}b^kc^l = b^jc^i. \end{aligned}\tag{6.56}$$

Multiplying equation 6.56 by $(a\cdot e_j)\,e_i$ gives

$$\begin{aligned}(a\cdot e_j)\,e_i\partial_{f_{ij}}\mathbf{f}(b)\cdot c &= (a\cdot e_j)\,e_ib^jc^i\\ &= (a\cdot e_j)\,cb^j\\ &= a_jb^jc = a_je^j\cdot e_kb^kc = (a\cdot b)\,c\end{aligned}\tag{6.57}$$

Since both $\mathbf{f}(b)\cdot c$ and $(a\cdot b)\,c$ do not depend upon the selection of the basis e_i, then $(a\cdot e_j)\,e_i\partial_{f_{ij}}$ also cannot depend upon the selection of the basis and we can define

$$\partial_{\mathbf{f}(a)} \equiv (a\cdot e_j)\,e_i\partial_{f_{ij}}\tag{6.58}$$

so that

$$\partial_{\mathbf{f}(a)}(\mathbf{f}(b)\cdot c) = (a\cdot b)\,c.\tag{6.59}$$

From equation 6.58 we see that $\partial_{\mathbf{f}(a)}$ is a vector operator and it obeys the product rule.

Now consider $\partial_{\mathbf{f}(a)}\langle\mathbf{f}(b\wedge c)\,B\rangle$ where $B=b_1\wedge b_2$ is a bivector. Then

$$\begin{aligned}\partial_{\mathbf{f}(a)}\langle\mathbf{f}(b\wedge c)\,B\rangle &= \partial_{\mathbf{f}(a)}\left((\mathbf{f}(b)\wedge\mathbf{f}(c))\cdot(b_1\wedge b_2)\right)\\ &= \dot{\partial}_{\dot{\mathbf{f}}(a)}\left(\left(\dot{\mathbf{f}}(b)\wedge\mathbf{f}(c)\right)\cdot(b_1\wedge b_2)+\left(\mathbf{f}(b)\wedge\dot{\mathbf{f}}(c)\right)\cdot(b_1\wedge b_2)\right)\\ &= \dot{\partial}_{\dot{\mathbf{f}}(a)}\left(\left(\dot{\mathbf{f}}(b)\wedge\mathbf{f}(c)\right)\cdot(b_1\wedge b_2)-\left(\dot{\mathbf{f}}(c)\wedge\mathbf{f}(b)\right)\cdot(b_1\wedge b_2)\right),\end{aligned}\tag{6.60}$$

but by equation B.10 (Appendix B)

$$(\mathbf{f}(b)\wedge\mathbf{f}(c))\cdot(b_1\wedge b_2) = (\mathbf{f}(b)\cdot b_2)(\mathbf{f}(c)\cdot b_1)-(\mathbf{f}(b)\cdot b_1)(\mathbf{f}(c)\cdot b_2)\tag{6.61}$$

so that (also using equation B.2)

$$\begin{aligned}\dot{\partial}_{\mathbf{f}(a)}\left(\left(\dot{\mathbf{f}}(b)\wedge\mathbf{f}(c)\right)\cdot(b_1\wedge b_2)\right) &= \dot{\partial}_{\mathbf{f}(a)}\left(\left(\dot{\mathbf{f}}(b)\cdot b_2\right)(\mathbf{f}(c)\cdot b_1)-\left(\dot{\mathbf{f}}(b)\cdot b_1\right)(\mathbf{f}(c)\cdot b_2)\right)\\ &= (a\cdot b)\left((\mathbf{f}(c)\cdot b_1)\,b_2-(\mathbf{f}(c)\cdot b_2)\,b_1\right)\\ &= (a\cdot b)\,\mathbf{f}(c)\cdot(b_1\wedge b_2) = (a\cdot b)\,(\mathbf{f}(c)\cdot B).\end{aligned}\tag{6.62}$$

Thus equation 6.60 becomes

$$\begin{aligned}\partial_{\mathbf{f}(a)}\left\langle \mathbf{f}\left(b\wedge c\right)B\right\rangle &= \left(a\cdot b\right)\left(\mathbf{f}\left(c\right)\cdot B\right)-\left(a\cdot c\right)\left(\mathbf{f}\left(b\right)\cdot B\right)\\ &= \left(\left(a\cdot b\right)\mathbf{f}\left(c\right)-\left(a\cdot c\right)\mathbf{f}\left(b\right)\right)\cdot B\\ &= \mathbf{f}\left(\left(a\cdot b\right)c-\left(a\cdot c\right)b\right)\cdot B\\ &= \mathbf{f}\left(a\cdot\left(b\wedge c\right)\right)\cdot B. \end{aligned}\tag{6.63}$$

In general if A_2 and B_2 are grade 2 multivectors then by linearity

$$\partial_{\mathbf{f}(a)}\left\langle \mathbf{f}\left(A_2\right)B_2\right\rangle = \mathbf{f}\left(a\cdot A_2\right)\cdot B_2.\tag{6.64}$$

The general case can be proved using grade analysis. First consider the following (where the subscript indicates the grade of the multivector and C is a general multivector)

$$\left\langle A_pC\right\rangle = \left\langle A_p\left(C_0+\cdots+C_n\right)\right\rangle\tag{6.65}$$

then the lowest grade of the general product term A_pC_q is $|p-q|$ which is only zero (a scalar) if $p=q$. Thus

$$\left\langle A_pC\right\rangle = \left\langle A_pC_p\right\rangle = A_p\cdot C_p\tag{6.66}$$

Thus we may write by applying equation 6.52 (C is the multivector $\left(\mathbf{f}\left(a_2\right)\dots\mathbf{f}\left(a_r\right)\right)B_r$)

$$\begin{aligned}\left\langle\left(\mathbf{f}\left(a_1\right)\wedge\dots\wedge\mathbf{f}\left(a_r\right)\right)B_r\right\rangle &= \left\langle\left(\mathbf{f}\left(a_1\right)\dots\mathbf{f}\left(a_r\right)\right)B_r\right\rangle\\ &= \mathbf{f}\left(a_1\right)\cdot\left\langle\mathbf{f}\left(a_2\right)\dots\mathbf{f}\left(a_r\right)B_r\right\rangle_1 \end{aligned}\tag{6.67}$$

so that

$$\begin{aligned}\dot{\partial}_{\mathbf{f}(a)}\left\langle\left(\dot{\mathbf{f}}\left(a_1\right)\wedge\dots\wedge\mathbf{f}\left(a_r\right)\right)B_r\right\rangle &= \dot{\partial}_{\mathbf{f}(a)}\left(\dot{\mathbf{f}}\left(a_1\right)\cdot\left\langle\mathbf{f}\left(a_2\right)\dots\mathbf{f}\left(a_r\right)B_r\right\rangle_1\right)\\ &= \left(a\cdot a_1\right)\left\langle\mathbf{f}\left(a_2\right)\dots\mathbf{f}\left(a_r\right)B_r\right\rangle_1\\ &= \left\langle\left(a\cdot a_1\right)\left(\mathbf{f}\left(a_2\right)\dots\mathbf{f}\left(a_r\right)\right)B_r\right\rangle_1\\ &= \left\langle\left(\mathbf{f}\left(\left(a\cdot a_1\right)a_2\right)\dots\mathbf{f}\left(a_r\right)\right)B_r\right\rangle_1\\ &= \left\langle\left(\mathbf{f}\left(\left(a\cdot a_1\right)a_2\right)\wedge\dots\wedge\mathbf{f}\left(a_r\right)\right)B_r\right\rangle_1\\ &= \left\langle\mathbf{f}\left(\left(a\cdot a_1\right)a_2\wedge\dots\wedge a_r\right)B_r\right\rangle_1. \end{aligned}\tag{6.68}$$

Thus (using equation A.5 in Appendix A)

$$\begin{aligned}\partial_{\mathbf{f}(a)}\left\langle\left(\mathbf{f}\left(a_1\right)\wedge\dots\wedge\mathbf{f}\left(a_r\right)\right)B_r\right\rangle &= \sum_{i=1}^{r}\left(-1\right)^{r+1}\left\langle\mathbf{f}\left(\left(a\cdot a_i\right)a_1\wedge\dots\wedge\breve{a}_i\wedge\dots\wedge a_r\right)B_r\right\rangle_1\\ &= \left\langle\mathbf{f}\left(\left(\sum_{i=1}^{r}\left(-1\right)^{r+1}a\cdot a_i\right)a_1\wedge\dots\wedge\breve{a}_i\wedge\dots\wedge a_r\right)B_r\right\rangle_1\\ &= \left\langle\mathbf{f}\left(a\cdot\left(a_1\wedge\dots\wedge a_r\right)\right)B_r\right\rangle_1. \end{aligned}\tag{6.69}$$

Using linearity the general case is

$$\partial_{\mathbf{f}(a)} \left\langle \mathbf{f}\left(A\right) B\right\rangle = \sum_r \left\langle \mathbf{f}\left(a \cdot \left\langle A\right\rangle_r\right) \left\langle B\right\rangle_r\right\rangle_1 . \tag{6.70}$$

For a fixed r-grade multivector A_r we can write

$$\begin{aligned} \left\langle \mathbf{f}\left(A_r\right) \dot{X}_r \right\rangle \dot{\partial}_{X_r} &= \left\langle \mathbf{f}\left(A_r\right) \frac{\partial X^{i_{\{r\}}}}{\partial X^{j_{\{r\}}}} e_{i_{\{r\}}} \right\rangle e^{j_{\{r\}}} \\ &= \left\langle \mathbf{f}\left(A_r\right) e_{i_{\{r\}}} \right\rangle e^{i_{\{r\}}} \\ &= \left(\mathbf{f}\left(A_r\right) \cdot e_{i_{\{r\}}}\right) e^{i_{\{r\}}} = \mathbf{f}\left(A_r\right). \end{aligned} \tag{6.71}$$

Applying equation 6.70 to equation 6.71 gives (let $\mathbf{f}\left(a \cdot A_r\right) = B_{i_1 \ldots i_{r-1}} e^{i_{r-1}} \wedge \ldots \wedge e^{i_1}$ be a coordinant expansion for $\mathbf{f}\left(a \cdot A_r\right)$)

$$\begin{aligned} \partial_{\mathbf{f}(a)} \mathbf{f}\left(A_r\right) &= \partial_{\mathbf{f}(a)} \left\langle \mathbf{f}\left(A_r\right) \dot{X}_r \right\rangle \dot{\partial}_{X_r} \\ &= \left(\mathbf{f}\left(a \cdot A_r\right) \cdot \dot{X}_r\right) \dot{\partial}_{X_r} \\ &= B_{i_1 \ldots i_{r-1}} \left(e^{i_{r-1}} \wedge \ldots \wedge e^{i_1}\right) \cdot \left(e_{j_1} \wedge \ldots \wedge e_{j_r}\right) e^{j_r} \wedge \ldots \wedge e^{j_1}. \end{aligned} \tag{6.72}$$

From what we know about subspaces the coefficients of $B_{i_1 \ldots i_{r-1}}$ are zero unless the $e_{j_1} \wedge \ldots \wedge e_{j_r}$ contain the same vectors (possibly in a different order) as the $e^{i_{r-1}} \wedge \ldots \wedge e^{i_1}$ plus one additional basis vector, e_j. Thus (let $E^{r-1} = e^{i_{r-1}} \wedge \ldots \wedge e^{i_1}$ and $E_{r-1} = e_{i_{r-1}} \wedge \ldots \wedge e_{i_1}$)

$$\left(e^{i_{r-1}} \wedge \ldots \wedge e^{i_1}\right) \cdot \left(e_{j_1} \wedge \ldots \wedge e_{j_r}\right) e^{j_r} \wedge \ldots \wedge e^{j_1} = \left(E^{r-1} \cdot \left(E_{r-1} \wedge e_j\right)\right) \left(e^j \wedge E^{r-1}\right). \tag{6.73}$$

The l.h.s of equation 6.73 can also be put in the form of the r.h.s. since even if the order of the e_{j_k}'s are scrambled they can alway be put into the same order as the e_{i_k}'s via transposition. If the order of the e^{j_k}'s are made to be the reverse of the e_{j_k}'s any minus signs generated will cancell out. Also the E_{r-1} and the E^{r-1} can be exchanged since they are scalar multiples of one another. The total number of possible e_j's is $n - r + 1$ where n is the dimension of the vector space. Now reduce the scalar coefficient of the blade in equation 6.73

$$\begin{aligned} \left(E^{r-1} \cdot \left(E_{r-1} \wedge e_j\right)\right) &= \left\langle E^{r-1} \left(E_{r-1} \wedge e_j\right)\right\rangle_1 \\ &= \left\langle E^{r-1} \left(E_{r-1} e_j - E_{r-1} \cdot e_j\right)\right\rangle_1 \\ &= e_j - \left\langle E_{r-1} \left(E^{r-1} \cdot e_j\right)\right\rangle_1 \\ &= e_j. \end{aligned} \tag{6.74}$$

Now reduce

$$\begin{aligned} e_j\left(e^j\wedge E^{r-1}\right) &= e_j\cdot\left(e^j\wedge E^{r-1}\right)+e_j\wedge\left(e^j\wedge E^{r-1}\right)\\ &= e_j\cdot\left(e^j\wedge E^{r-1}\right)\\ &= \left(e_j\cdot e^j\right)E^{r-1}+\sum_{l=1}^{r-1}(-1)^l\left(e_j\cdot e^{i_l}\right)e^{i_{r-1}}\wedge\ldots\wedge\breve{e}^{i_l}\wedge\ldots\wedge e^{i_1}\\ &= E^{r-1}. \end{aligned} \tag{6.75}$$

Thus

$$\partial_{\mathbf{f}(a)}\mathbf{f}\left(A_r\right)=\left(n-r+1\right)\mathbf{f}\left(a\cdot A_r\right). \tag{6.76}$$

Now let $A_r=I$ then

$$\partial_{\mathbf{f}(a)}=\partial_{\mathbf{f}(a)}\det\left(\mathbf{f}\right)I=\mathbf{f}\left(a\cdot I\right), \tag{6.77}$$

but by equation 1.94 we have

$$\begin{aligned} \det\left(\mathbf{f}\right)\mathbf{f}^{-1}\left(a\right) &= I\bar{\mathbf{f}}\left(I^{-1}a\right)\\ &= I^{-1}\bar{\mathbf{f}}\left(Ia\right)\\ \det\left(\mathbf{f}\right)I\mathbf{f}^{-1}\left(a\right) &= \bar{\mathbf{f}}\left(Ia\right)\\ \det\left(\mathbf{f}\right)\left(\mathbf{f}^{-1}\left(a\right)\right)^{\dagger}I^{\dagger} &= \bar{\mathbf{f}}\left(\left(Ia\right)^{\dagger}\right)\\ \det\left(\mathbf{f}\right)\mathbf{f}^{-1}\left(a\right)I^{\dagger} &= \bar{\mathbf{f}}\left(aI^{\dagger}\right)\\ \det\left(\mathbf{f}\right)\mathbf{f}^{-1}\left(a\right)I &= \bar{\mathbf{f}}\left(aI\right)\\ &= \bar{\mathbf{f}}\left(a\cdot I\right)\\ \det\left(\mathbf{f}\right)\bar{\mathbf{f}}^{-1}\left(a\right)I &= \mathbf{f}\left(a\cdot I\right) \end{aligned} \tag{6.78}$$

or

$$\partial_{\mathbf{f}(a)}\det\left(\mathbf{f}\right)=\det\left(\mathbf{f}\right)\bar{\mathbf{f}}^{-1}\left(a\right). \tag{6.79}$$

Equation 6.69 can also be used to calculate the functional derivative of the adjoint. The result for $r>1$ is

$$\begin{aligned} \partial_{\mathbf{f}(a)}\bar{\mathbf{f}}\left(A_r\right) &= \partial_{\mathbf{f}(a)}\left\langle\mathbf{f}\left(\dot{X}_r\right)A_r\right\rangle\dot{\partial}_{X_r}\\ &= \mathbf{f}\left(a\cdot\dot{X}_r\right)\cdot A_r\partial_{X_r}. \end{aligned} \tag{6.80}$$

Equation 6.80 cannot be used when $r=1$ since $\mathbf{f}\left(a\cdot X_1\right)$ is not defined. Let $A_r=b$ then using components we have

$$\begin{aligned}
\partial_{\mathbf{f}(a)}\bar{\mathbf{f}}\left(b\right) &= a_j e_i \partial_{f_{ij}} e^k \bar{f}_{kl} b^l \\
&= a_j e_i \partial_{f_{ij}} e^k f_{lk} b^l \\
&= a_j e_i \delta^i_l \delta^j_k e^k b^l \\
&= a_j e_i \mathrm{e}^j b^i = ba. \qquad (6.81)
\end{aligned}$$

Chapter 7

Multilinear Functions (Tensors)

A multivector multilinear function[1] is a multivector function $T(A_1, \dots, A_r)$ that is linear in each of its arguments[2] The tensor could be non-linearly dependent on a set of additional arguments such as the position coordinates x^i in the case of a tensor field defined on a manifold. If x denotes the coordinate tuple for a manifold we denote the dependence of T on x by $T(A_1, \dots, A_r; x)$.

T is a *tensor* of degree r if each variable $A_j \in \mathcal{V}_n$ ($\mathcal{V}_n$ is an n-dimensional vector space). More generally if each $A_j \in \mathcal{G}(\mathcal{V}_n)$ (the geometric algebra of $\mathcal{V}_n$), we call T an *extensor* of degree-r on $\mathcal{G}(\mathcal{V}_n)$.

If the values of $T(a_1, \dots, a_r)$ ($a_j \in \mathcal{V}_n \; \forall \; 1 \le j \le r$) are s-vectors (pure grade s multivectors in $\mathcal{G}(\mathcal{V}_n)$) we say that T has grade s and rank $r+s$. A tensor of grade zero is called a *multilinear form*.

In the normal definition of tensors as multilinear functions the tensor is defined as a multilinear mapping

$$T : \bigtimes_{i=1}^{r} \mathcal{V}_n \rightarrow \Re,$$

so that the standard tensor definition is an example of a grade zero degree/rank r tensor in our definition.

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[1] We are following the treatment of Tensors in section 3–10 of [4].

[2] We assume that the arguments are elements of a vector space or more generally a geometric algebra so that the concept of linearity is meaningful.

7.1 Algebraic Operations

The properties of tensors are ($\alpha \in \Re$, $a_j, b \in \mathcal{V}_n$, T and S are tensors of rank r, and $\circ$ is any multivector multiplicative operation)

$$
\begin{aligned}
T\left(a_1, \ldots, \alpha a_j, \ldots, a_r\right) =& \alpha T\left(a_1, \ldots, a_j, \ldots, a_r\right), && (7.1)\\
T\left(a_1, \ldots, a_j + b, \ldots, a_r\right) =& T\left(a_1, \ldots, a_j, \ldots, a_r\right) + T\left(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_r\right), && (7.2)\\
(T \pm S)\left(a_1, \ldots, a_r\right) \equiv& T\left(a_1, \ldots, a_r\right) \pm S\left(a_1, \ldots, a_r\right). && (7.3)
\end{aligned}
$$

Now let T be of rank r and S of rank s then the product of the two tensors is

$$
(T \circ S)\left(a_1, \ldots, a_{r+s}\right) \equiv T\left(a_1, \ldots, a_r\right) \circ S\left(a_{r+1}, \ldots, a_{r+s}\right), \tag{7.4}
$$

where "$\circ$" is any multivector multiplicative operation.

7.2 Covariant, Contravariant, and Mixed Representations

The arguments (vectors) of the multilinear fuction can be represented in terms of the basis vectors or the reciprocal basis vectors[3]

$$
\begin{aligned}
a_j =& a^{i_j} \boldsymbol{e}_{i_j}, && (7.5)\\
=& a_{i_j} \boldsymbol{e}^{i_j}. && (7.6)
\end{aligned}
$$

Equation (7.5) gives a_j in terms of the basis vectors and eq (7.6) in terms of the reciprocal basis vectors. The index j refers to the argument slot and the indices i_j the components of the vector in terms of the basis. The Einstein summation convention is used throughout. The covariant representation of the tensor is defined by

$$
\begin{aligned}
T_{i_1 \ldots i_r} \equiv& T\left(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}\right) && (7.7)\\
T\left(a_1, \ldots, a_r\right) =& T\left(a^{i_1} \boldsymbol{e}_{i_1}, \ldots, a^{i_r} \boldsymbol{e}_{i_r}\right)\\
=& T\left(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}\right) a^{i_1} \ldots a^{i_r}\\
=& T_{i_1 \ldots i_r} a^{i_1} \ldots a^{i_r}. && (7.8)
\end{aligned}
$$

---

[3] When the a_j vectors are expanded in terms of a basis we need a notatation that lets one determine which vector argument, j, the scalar components are associated with. Thus when we expand the vector in terms of the basis we write $a_j = a^{i_j} \boldsymbol{e}_{i_j}$ with the Einstein summation convention applied over the i_j indices. In the expansion the j in the a^{i_j} determines which argument in the tensor function the a^{i_j} coefficients are associated with. Thus it is always the subscript of the component super or subscript that determines the argument the coefficient is associated with.

Likewise for the contravariant representation

$$\begin{aligned}
T^{i_1\ldots i_r} &\equiv T\left(\boldsymbol{e}^{i_1},\ldots,\boldsymbol{e}^{i_r}\right) && (7.9)\\
T\left(a_1,\ldots,a_r\right) &= T\left(a_{i_1}\boldsymbol{e}^{i_1},\ldots,a_{i_r}\boldsymbol{e}^{i_r}\right) &&\\
&= T\left(\boldsymbol{e}^{i_1},\ldots,\boldsymbol{e}^{i_r}\right)a_{i_1}\ldots a_{i_r} &&\\
&= T^{i_1\ldots i_r}a_{i_1}\ldots a_{i_r}. && (7.10)
\end{aligned}$$

One could also have a mixed representation

$$\begin{aligned}
T_{i_1\ldots i_s}{}^{i_{s+1}\ldots i_r} &\equiv T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{i_s},\boldsymbol{e}^{i_{s+1}}\ldots\boldsymbol{e}^{i_r}\right) && (7.11)\\
T\left(a_1,\ldots,a_r\right) &= T\left(a^{i_1}\boldsymbol{e}_{i_1},\ldots,a^{i_s}\boldsymbol{e}_{i_s},a_{i_{s+1}}\boldsymbol{e}^{i_s},\ldots,a_{i_r}\boldsymbol{e}^{i_r}\right) &&\\
&= T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{i_s},\boldsymbol{e}^{i_{s+1}},\ldots,\boldsymbol{e}^{i_r}\right)a^{i_1}\ldots a^{i_s}a_{i_{s+1}}\ldots a_{i_r} &&\\
&= T_{i_1\ldots i_s}{}^{i_{s+1}\ldots i_r}a^{i_1}\ldots a^{i_s}a_{i_{s+1}},\ldots a_{i_r}. && (7.12)
\end{aligned}$$

In the representation of T one could have any combination of covariant (lower) and contravariant (upper) indices.

To convert a covariant index to a contravariant index simply consider

$$\begin{aligned}
T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}^{i_j},\ldots,\boldsymbol{e}_{i_r}\right) &= T\left(\boldsymbol{e}_{i_1},\ldots,g^{i_jk_j}\boldsymbol{e}_{k_j},\ldots,\boldsymbol{e}_{i_r}\right) &&\\
&= g^{i_jk_j}T\left(\boldsymbol{e}_{i_1},\ldots,\boldsymbol{e}_{k_j},\ldots,\boldsymbol{e}_{i_r}\right) &&\\
T_{i_1\ldots}{}^{i_j}{}_{\ldots i_r} &= g^{i_jk_j}T_{i_1\ldots i_j\ldots i_r}. && (7.13)
\end{aligned}$$

Similarly one could raise a lower index with $g_{i_jk_j}$.

7.3 Contraction

The contraction of a tensor between the j^{th} and k^{th} variables (slots) is[4]

$$T\left(a_i,\dots,a_{j-1},\nabla_{a_k},a_{j+1},\dots,a_r\right)=\nabla_{a_j}\cdot\left(\nabla_{a_k}T\left(a_1,\dots,a_r\right)\right). \tag{7.14}$$

This operation reduces the rank of the tensor by two. This definition gives the standard results for *metric contraction* which is proved as follows for a rank r grade zero tensor (the circumflex "$\breve{}$" indicates that a term is to be deleted from the product).

$$\begin{aligned}
T\left(a_1,\dots,a_r\right) =& a^{i_1}\dots a^{i_r}T_{i_1\dots i_r} && (7.15)\\
\nabla_{a_j}T =& \boldsymbol{e}^{l_j}a^{i_1}\dots\left(\partial_{a^{l_j}}a^{i_j}\right)\dots a_{i_r}T_{i_1\dots i_r}\\
=& \boldsymbol{e}^{l_j}\delta_{l_j}^{i_j}a^{i_1}\dots\breve{a}^{i_j}\dots a^{i_r}T_{i_1\dots i_r} && (7.16)\\
\nabla_{a_m}\cdot\left(\nabla_{a_j}T\right) =& \boldsymbol{e}^{k_m}\cdot\boldsymbol{e}^{l_j}\delta_{l_j}^{i_j}a^{i_1}\dots\breve{a}^{i_j}\dots\left(\partial_{a^{k_m}}a^{i_m}\right)\dots a^{i_r}T_{i_1\dots i_r}\\
=& g^{k_m l_j}\delta_{l_j}^{i_j}\delta_{k_m}^{i_m}a^{i_1}\dots\breve{a}^{i_j}\dots\breve{a}^{i_m}\dots a^{i_r}T_{i_1\dots i_r}\\
=& g^{i_m i_j}a^{i_1}\dots\breve{a}^{i_j}\dots\breve{a}^{i_m}\dots a^{i_r}T_{i_1\dots i_j\dots i_m\dots i_r}\\
=& g^{i_j i_m}a^{i_1}\dots\breve{a}^{i_j}\dots\breve{a}^{i_m}\dots a^{i_r}T_{i_1\dots i_j\dots i_m\dots i_r}\\
=& \left(g^{i_j i_m}T_{i_1\dots i_j\dots i_m\dots i_r}\right)a^{i_1}\dots\breve{a}^{i_j}\dots\breve{a}^{i_m}\dots a^{i_r} && (7.17)
\end{aligned}$$

Equation (7.17) is the correct formula for the metric contraction of a tensor.

If we have a mixed representation of a tensor, $T_{i_1\dots}{}^{i_j}{}_{\dots i_k\dots i_r}$, and wish to contract between an upper and lower index (i_j and i_k) first lower the upper index and then use eq (7.17) to contract the result. Remember lowering the index does *not* change the tensor, only the *representation* of the tensor, while contraction results in a *new* tensor. First lower index

$$T_{i_1\dots}{}^{i_j}{}_{\dots i_k\dots i_r}\xrightarrow{\text{Lower Index}}g_{i_jk_j}T_{i_1\dots}{}^{k_j}{}_{\dots i_k\dots i_r} \tag{7.18}$$

---

[4] The notation of the l.h.s. of eq (7.14) is new and is defined by $\nabla_{a_k}=\boldsymbol{e}^{l_k}\partial_{a^{l_k}}$ and (the assumption of the notation is that the $\partial_{a^{l_k}}$ can be factored out of the argument like a simple scalar)

$$\begin{aligned}
T\left(a_i,\dots,a_{j-1},\nabla_{a_k},a_{j+1},\dots,a_r\right) \equiv& T\left(a_i,\dots,a_{j-1},\boldsymbol{e}^{l_k}\partial_{a^{l_k}},a_{j+1},\dots,a^{i_k}\boldsymbol{e}_{i_k},\dots,a_r\right)\\
=& T\left(a_i,\dots,a_{j-1},\boldsymbol{e}_{j_k}g^{j_kl_k}\partial_{a^{l_k}},a_{j+1},\dots,a^{i_k}\boldsymbol{e}_{i_k},\dots,a_r\right)\\
=& g^{j_kl_k}\partial_{a^{l_k}}a^{i_k}T\left(a_i,\dots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\dots,\boldsymbol{e}_{i_k},\dots,a_r\right)\\
=& g^{j_kl_k}\delta_{l_k}^{i_k}T\left(a_i,\dots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\dots,\boldsymbol{e}_{i_k},\dots,a_r\right)\\
=& g^{j_ki_k}T\left(a_i,\dots,a_{j-1},\boldsymbol{e}_{j_k},a_{j+1},\dots,\boldsymbol{e}_{i_k},\dots,a_r\right)\\
=& g^{j_ki_k}T_{i_1\dots i_{j-1}j_ki_{j+1}\dots i_k\dots i_r}a^{i_1}\dots\breve{a}^{i_j}\dots\breve{a}^{i_k}\dots a^{i_r}.
\end{aligned}$$

Now contract between i_j and i_k and use the properties of the metric tensor.

$$\begin{aligned} g_{i_j k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} \xRightarrow{\text{Contract}} & g^{i_j i_k} g_{i_j k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} \\ =& \delta^{i_k}_{k_j} T_{i_1 \ldots}{}^{k_j}{}_{\ldots i_k \ldots i_r} . \end{aligned} \tag{7.19}$$

Equation (7.19) is the standard formula for contraction between upper and lower indices of a mixed tensor.

7.4 Differentiation

If $T(a_1, \ldots, a_r; x)$ is a tensor field (a function of the position vector, x, for a vector space or the coordinate tuple, x, for a manifold) the tensor directional derivative is defined as

$$\mathcal{D}T(a_1, \ldots, a_r; x) \equiv (a_{r+1} \cdot \nabla) T(a_1, \ldots, a_r; x), \tag{7.20}$$

assuming the a^{i_j} coefficients are not a function of the coordinates.

This gives for a grade zero rank r tensor

$$\begin{aligned} (a_{r+1} \cdot \nabla) T(a_1, \ldots, a_r) =& a^{i_{r+1}} \partial_{x^{i_{r+1}}} a^{i_1} \ldots a^{i_r} T_{i_1 \ldots i_r}, \\ =& a^{i_1} \ldots a^{i_r} a^{i_{r+1}} \partial_{x^{i_{r+1}}} T_{i_1 \ldots i_r}. \end{aligned} \tag{7.21}$$

7.5 From Vector/Multivector to Tensor

A rank one tensor corresponds to a vector since it satisfies all the axioms for a vector space, but a vector in not necessarily a tensor since not all vectors are multilinear (actually in the case of vectors a linear function) functions. However, there is a simple isomorphism between vectors and rank one tensors defined by the mapping $v(a) : \mathcal{V} \rightarrow \Re$ such that if $v, a \in \mathcal{V}$

$$v(a) \equiv v \cdot a. \tag{7.22}$$

So that if $v = v^i \boldsymbol{e}_i = v_i \boldsymbol{e}^i$ the covariant and contravariant representations of v are (using $\boldsymbol{e}^i \cdot \boldsymbol{e}_j = \delta^i_j$)

$$v(a) = v_i a^i = v^i a_i. \tag{7.23}$$

The equivalent mapping from a pure r-grade multivector A to a rank-r tensor $A(a_1, \ldots, a_r)$ is

$$A(a_1, \ldots, a_r) = A \cdot (a_1 \wedge \ldots \wedge a_r). \tag{7.24}$$

Note that since the sum of two tensor of different ranks is not defined we cannot represent a spinor with tensors. Additionally, even if we allowed for the summation of tensors of different ranks we would also have to redefine the tensor product to have the properties of the geometric wedge product. Likewise, multivectors can only represent completely antisymmetric tensors of rank less than or equal to the dimension of the base vector space.

7.6 Parallel Transport Definition and Example

The defintion of parallel transport is that if a and b are tangent vectors in the tangent spaced of the manifold then

$$\left(a\cdot\nabla_x\right)b = 0 \tag{7.25}$$

if b is parallel transported in the direction of a (infinitesimal parallel transport). Since $b = b^i\boldsymbol{e}_i$ and the derivatives of $\boldsymbol{e}_i$ are functions of the x^i's then the b^i's are also functions of the x^i's so that in order for eq (7.25) to be satisfied we have

$$\begin{aligned}
\left(a\cdot\nabla_x\right)b =& a^i\partial_{x^i}\left(b^j\boldsymbol{e}_j\right) \\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^j\partial_{x^i}\boldsymbol{e}_j\right) \\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^j\Gamma^k_{ij}\boldsymbol{e}_k\right) \\
=& a^i\left(\left(\partial_{x^i}b^j\right)\boldsymbol{e}_j + b^k\Gamma^j_{ik}\boldsymbol{e}_j\right) \\
=& a^i\left(\left(\partial_{x^i}b^j\right) + b^k\Gamma^j_{ik}\right)\boldsymbol{e}_j = 0.
\end{aligned} \tag{7.26}$$

Thus for b to be parallel transported (infinitesimal parallel transport in any direction a) we must have

$$\partial_{x^i}b^j = -b^k\Gamma^j_{ik}. \tag{7.27}$$

The geometric meaning of parallel transport is that for an infinitesimal rotation and dilation of the basis vectors (cause by infinitesimal changes in the x^i's) the direction and magnitude of the vector b does not change to first order.

If we apply eq (7.27) along a parametric curve defined by $x^j\left(s\right)$ we have

$$\begin{aligned}
\frac{db^j}{ds} =& \frac{dx^i}{ds}\frac{\partial b^j}{\partial x^i} \\
=& -b^k\frac{dx^i}{ds}\Gamma^j_{ik},
\end{aligned} \tag{7.28}$$

and if we define the initial conditions $b^j(0)\,\boldsymbol{e}_j$. Then eq (7.28) is a system of first order linear differential equations with intial conditions and the solution, $b^j(s)\,\boldsymbol{e}_j$, is the parallel transport of the vector $b^j(0)\,\boldsymbol{e}_j$.

An equivalent formulation for the parallel transport equation is to let $\gamma(s)$ be a parametric curve in the manifold defined by the tuple $\gamma(s) = \left(x^1(s), \dots, x^n(s)\right)$. Then the tangent to $\gamma(s)$ is given by

$$\frac{d\gamma}{ds} \equiv \frac{dx^i}{ds}\boldsymbol{e}_i \tag{7.29}$$

and if $v(x)$ is a vector field on the manifold then

$$\begin{aligned}\left(\frac{d\gamma}{ds}\cdot\nabla_x\right)v =& \frac{dx^i}{ds}\frac{\partial}{\partial x^i}\left(v^j\boldsymbol{e}_j\right)\\ =& \frac{dx^i}{ds}\left(\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + v^j\frac{\partial\boldsymbol{e}_j}{\partial x^i}\right)\\ =& \frac{dx^i}{ds}\left(\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + v^j\Gamma^k_{ij}\boldsymbol{e}_k\right)\\ =& \frac{dx^i}{ds}\frac{\partial v^j}{\partial x^i}\boldsymbol{e}_j + \frac{dx^i}{ds}v^k\Gamma^j_{ik}\boldsymbol{e}_j\\ =& \left(\frac{dv^j}{ds} + \frac{dx^i}{ds}v^k\Gamma^j_{ik}\right)\boldsymbol{e}_j\\ =& 0.\end{aligned} \tag{7.30}$$

Thus eq (7.30) is equivalent to eq (7.28) and parallel transport of a vector field along a curve is equivalent to the directional derivative of the vector field in the direction of the tangent to the curve being zero.

As a specific example of parallel transport consider a spherical manifold with a series of concentric circular curves and paralle transport a vector along each curve. Note that the circular curves are defined by

$$\begin{aligned}u(s) =& u_0 + a\cos\left(\frac{s}{2\pi a}\right)\\ v(s) =& v_0 + a\sin\left(\frac{s}{2\pi a}\right)\end{aligned}$$

where u and v are the manifold coordinates. The spherical manifold is defined by

$$\begin{aligned}x =& \cos(u)\,cos(v)\\ y =& \cos(u)\,sin(v)\\ z =& \sin(u)\,.\end{aligned}$$

Note that due to the dependence of the metric on the coordinates circles do not necessarily appear to be circular in the plots depending on the values of u_0 and v_0 (see fig 7.2). For symmetrical circles we have fig 7.1 and for asymmetrical circles we have fig 7.2. Note that the appearance of the transported (black) vectors is an optical illusion due to the projection. If the sphere were rotated we would see that the transported vectors are in the tangent space of the manifold.

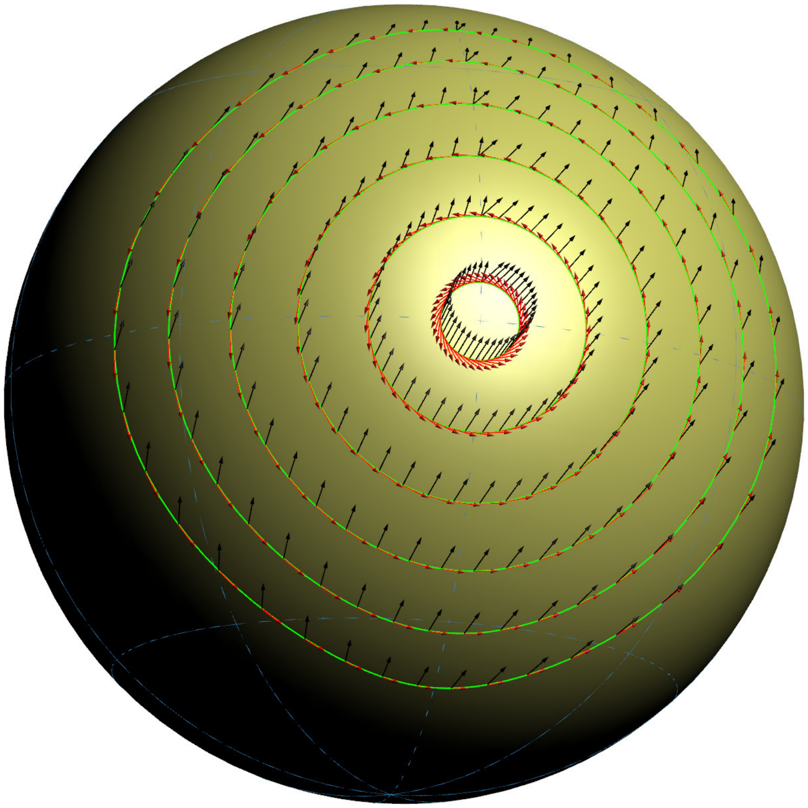

Figure 7.1: Parallel transport for $u_0 = 0$ and $v_0 = 0$. Red vectors are tangents to circular curves and black vectors are the vectors being transported.

If $\gamma(s) = (u(s), v(s))$ defines the transport curve then

$$\frac{d\gamma}{ds} = \frac{du}{ds}\boldsymbol{e}_u + \frac{dv}{ds}\boldsymbol{e}_v \tag{7.31}$$

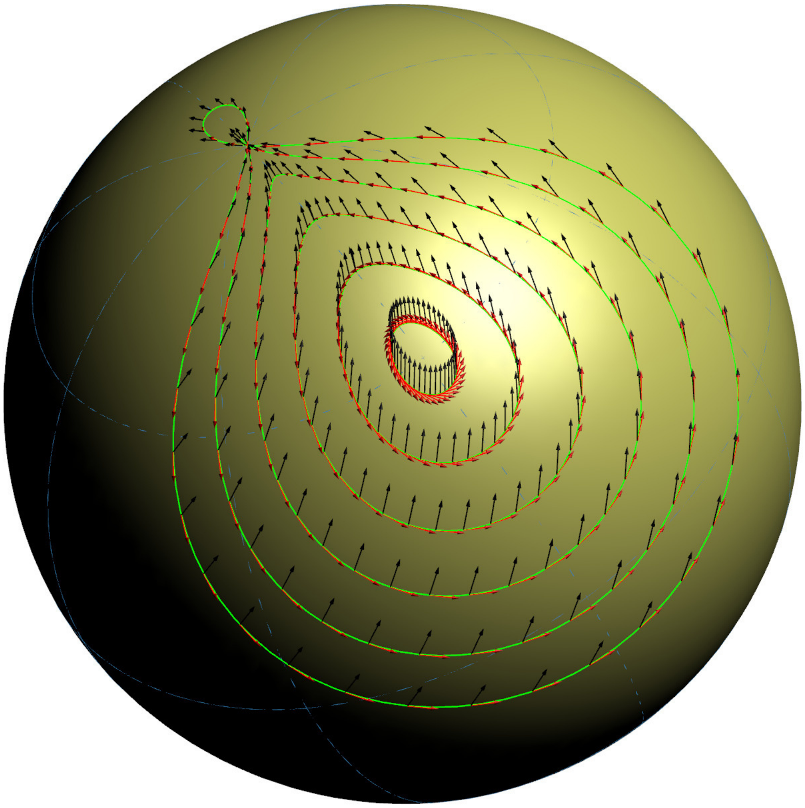

Figure 7.2: Parallel transport for $u_0 = \pi/4$ and $v_0 = \pi/4$. Red vectors are tangents to circular curves and black vectors are the vectors being transported.

and the transport equations are

$$\begin{aligned}
\left(\frac{d\gamma}{ds}\cdot\nabla\right) f =& \left(\frac{du}{ds}\frac{\partial f^u}{\partial u} + \frac{dv}{ds}\frac{\partial f^u}{\partial v} - \sin\left(u\right)\cos\left(u\right)\frac{dv}{ds}f^v\right)\boldsymbol{e}_u + \\
&\left(\frac{du}{ds}\frac{\partial f^v}{\partial u} + \frac{dv}{ds}\frac{\partial f^v}{\partial v} + \frac{\cos\left(u\right)}{\sin\left(u\right)}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right)\right)\boldsymbol{e}_u \\
=& \left(\frac{df^u}{ds} - \sin\left(u\right)\cos\left(u\right)\frac{dv}{ds}f^v\right)\boldsymbol{e}_u + \\
&\left(\frac{df^v}{ds} + \frac{\cos\left(u\right)}{\sin\left(u\right)}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right)\right)\boldsymbol{e}_u = 0 \qquad (7.32)
\end{aligned}$$

$$\frac{df^u}{ds} = \sin\left(u\right)\cos\left(u\right)\frac{dv}{ds}f^v \qquad (7.33)$$

$$\frac{df^v}{ds} = -\frac{\cos\left(u\right)}{\sin\left(u\right)}\left(\frac{du}{ds}f^v + \frac{dv}{ds}f^u\right) \qquad (7.34)$$

If the tensor component representation is contra-variant (superscripts instead of subscripts) we must use the covariant component representation of the vector arguements of the tensor, $a = a_i\boldsymbol{e}^i$. Then the definition of parallel transport gives

$$\begin{aligned}(a\cdot\nabla_x)\,b =& a^i\partial_{x^i}\left(b_j\boldsymbol{e}^j\right)\\ =& a^i\left(\left(\partial_{x^i}b_j\right)\boldsymbol{e}^j + b_j\partial_{x^i}\boldsymbol{e}^j\right),\end{aligned} \tag{7.35}$$

and we need

$$\left(\partial_{x^i}b_j\right)\boldsymbol{e}^j + b_j\partial_{x^i}\boldsymbol{e}^j = 0. \tag{7.36}$$

To satisfy equation (7.36) consider the following

$$\begin{aligned}\partial_{x^i}\left(\boldsymbol{e}^j\cdot\boldsymbol{e}_k\right) =& 0\\ \left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k + \boldsymbol{e}^j\cdot\left(\partial_{x^i}\boldsymbol{e}_k\right) =& 0\\ \left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k + \boldsymbol{e}^j\cdot\boldsymbol{e}_l\Gamma^l_{ik} =& 0\\ \left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k + \delta^j_l\Gamma^l_{ik} =& 0\\ \left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k + \Gamma^j_{ik} =& 0\\ \left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k =& -\Gamma^j_{ik}\end{aligned} \tag{7.37}$$

Now dot eq (7.36) into $\boldsymbol{e}_k$ giving[5]

$$\begin{aligned}\left(\partial_{x^i}b_j\right)\boldsymbol{e}^j\cdot\boldsymbol{e}_k + b_j\left(\partial_{x^i}\boldsymbol{e}^j\right)\cdot\boldsymbol{e}_k =& 0\\ \left(\partial_{x^i}b_j\right)\delta^k_j - b_j\Gamma^j_{ik} =& 0\\ \left(\partial_{x^i}b_k\right) =& b_j\Gamma^j_{ik}.\end{aligned} \tag{7.39}$$

7.7 Covariant Derivative of Tensors

The covariant derivative of a tensor field $T(a_1,\dots,a_r;x)$ (x is the coordinate tuple of which T can be a non-linear function) in the direction a_{r+1} is (remember $a_j = a^{k_j}\boldsymbol{e}_{k_j}$ and the $\boldsymbol{e}_{k_j}$ can be functions of x) the directional derivative of $T(a_1,\dots,a_r;x)$ where all the a_i vector arguments of T are parallel transported.

---

[5] These equations also show that

$$\partial_{x^i}\boldsymbol{e}^j = -\Gamma^j_{ik}\boldsymbol{e}^k. \tag{7.38}$$

Thus if we have a mixed representation of a tensor

$$T\left(a_1,\dots,a_r;x\right) = T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}\left(x\right)a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}, \tag{7.40}$$

the covariant derivative of the tensor is

$$\begin{aligned}
\left(a_{r+1}\cdot D\right)T\left(a_1,\dots,a_r;x\right) =& \frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}} \\
&+ \sum_{p=1}^{s}\frac{\partial a^{i_p}}{\partial x^{i_{r+1}}}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots\breve{a}^{i_p}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}} \\
&+ \sum_{q=s+1}^{r}\frac{\partial a_{i_p}}{\partial x^{i_{r+1}}}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots\breve{a}_{i_q}\dots a_{i_r}a^{i_{r+1}} \\
=& \frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}^{r}a^{i_{r+1}} \\
&- \sum_{p=1}^{s}\Gamma^{i_p}_{i_{r+1}l_p}T_{i_1\dots i_p\dots i_s}{}^{i_{s+1}\dots i_r}a^{i_1}\dots a^{l_p}\dots a^{i_s}a_{i_{s+1}}\dots a_{i_r}a^{i_{r+1}} \\
&+ \sum_{q=s+1}^{r}\Gamma^{l_q}_{i_{r+1}i_q}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_q\dots i_r}a^{i_1}\dots a^{i_s}a_{i_{s+1}}\dots a_{l_q}\dots a_{i_r}a^{i_{r+1}}.
\end{aligned} \tag{7.41}$$

From eq (7.41) we obtain the components of the covariant derivative to be

$$\frac{\partial T_{i_1\dots i_s}{}^{i_{s+1}\dots i_r}}{\partial x^{r+1}} - \sum_{p=1}^{s}\Gamma^{i_p}_{i_{r+1}l_p}T_{i_1\dots i_p\dots i_s}{}^{i_{s+1}\dots i_r} + \sum_{q=s+1}^{r}\Gamma^{l_q}_{i_{r+1}i_q}T_{i_1\dots i_s}{}^{i_{s+1}\dots i_q\dots i_r}. \tag{7.42}$$

To extend the covariant derivative to tensors with multivector values in the tangent space (geometric algebra of the tangent space) we start with the coordinate free definition of the covariant derivative of a conventional tensor using the following notation. Let $T\left(a_1,\dots,a_r;x\right)$ be a conventional tensor then the directional covariant derivative is

$$\left(b\cdot D\right)T = a^{i_1}\dots a^{i_r}\left(b\cdot\nabla\right)T\left(e_{i_1},\dots,e_{i_r};x\right) - \sum_{j=1}^{r}T\left(a_1,\dots,\left(b\cdot\nabla\right)a_j,\dots,a_r;x\right). \tag{7.43}$$

The first term on the r.h.s. of eq (7.44) is the directional derivative of T if we assume that the component coefficients of each of the a_j does not change if the coordinate tuple changes. The remaining terms in eq (7.44) insure that for the totality of eq (7.44) the directional derivative $\left(b\cdot\nabla\right)T$ is the same as that when all the a_j vectors are parallel transported. If in eq (7.44) we let $b \cdot \nabla$ be the directional derivative for a multivector field we have generalized the definintion of covariant derivative to include the cases where $T(a_1, \ldots, a_r; x)$ is a multivector and not only a scalar. Basically in eq (7.44) the terms $T(e_{i_1}, \ldots, e_{i_r}; x)$ are multivector fields and $(b \cdot \nabla) T(e_{i_1}, \ldots, e_{i_r}; x)$ is the direction derivative of each of the multivector fields that make up the component representation of the multivector tensor. The remaining terms in eq (7.44) take into account that for parallel transport of the a_i's the coefficients a^{i_j} are implicit functions of the coordinates x^k. If we define the symbol ∇_x to only refer to taking the geometric derivative with respect to an explicit dependence on the x coordinate tuple we can recast eq (7.44) into

$$(b \cdot D) T = (b \cdot \nabla_x) T(a_1, \ldots, a_r; x) - \sum_{j=1}^{r} T(a_1, \ldots, (b \cdot \nabla) a_j, \ldots, a_r; x). \tag{7.44}$$

7.8 Coefficient Transformation Under Change of Variable

In the previous sections on tensors a transformation of coordinate tuples $\bar{x}(x) = (\bar{x}^i(x), \ldots, \bar{x}^n(x))$, where $x = (x^1, \ldots, x^n)$, is not mentioned since the definition of a tensor as a multilinear function is invariant to the representation of the vectors (coordinate system). From our tensor definitions the effect of a coordinate transformation on the tensor components is simply calculated.

If $R(x) = R(\bar{x})$ is the defining vector function for a vector manifold (R is in the embedding space of the manifold) then[6]

$$\boldsymbol{e}_i = \frac{\partial R}{\partial x^i} = \frac{\partial \bar{x}^j}{\partial x^i} \bar{\boldsymbol{e}}_j \tag{7.45}$$

$$\bar{\boldsymbol{e}}_i = \frac{\partial R}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_j. \tag{7.46}$$

---

[6] For an abstract manifold the equation $\bar{\boldsymbol{e}}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} \boldsymbol{e}_j$ can be used as an defining relationship.

Thus we have

$$
\begin{aligned}
T\left(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_1}\right) =& T_{i_1 \ldots i_r} && (7.47)\\
T\left(\bar{\boldsymbol{e}}_{j_1}, \ldots, \bar{\boldsymbol{e}}_{j_1}\right) =& \bar{T}_{j_1 \ldots j_r} && (7.48)\\
T\left(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_1}\right) =& T\left(\frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \bar{\boldsymbol{e}}_{j_1}, \ldots, \frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}} \bar{\boldsymbol{e}}_{j_1}\right) && (7.49)\\
=& \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \ldots \frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}} T\left(\bar{\boldsymbol{e}}_{j_1}, \ldots, \bar{\boldsymbol{e}}_{j_1}\right) && (7.50)\\
T_{i_1 \ldots i_r} =& \frac{\partial \bar{x}^{j_1}}{\partial x^{i_1}} \ldots \frac{\partial \bar{x}^{j_r}}{\partial x^{i_r}} \bar{T}_{j_1 \ldots jr}. && (7.51)
\end{aligned}
$$

Equation (7.51) is the standard formula for the transformation of tensor components.

Chapter 8

Lagrangian and Hamiltonian Methods[1]

8.1 Lagrangian Theory for Discrete Systems

8.1.1 The Euler-Lagrange Equations

Let a system be described by multivector variables X_i, $i = 1, \dots, m$. The Lagrangian L is a scalar valued function of the X_i, $\dot{X}_i$ (here the dot refers to the time derivative), and possibly the time, t. The action for the system, S, over a time interval is given by the integral

$$S \equiv \int_{t_1}^{t_2} dtL\left(X_i, \dot{X}_i, t\right). \tag{8.1}$$

The statement of the principal of least action is that the variation of the action $\delta S = 0$. The rigorous definition of $\delta S = 0$ is let

$$X_i'(t) = X_i(t) + \epsilon Y_i(t) \tag{8.2}$$

where $Y_i(t)$ is an arbitrary differentiable multivector function of time except that $Y_i(t_1) = Y_i(t_2) = 0$. Then

$$\delta S \equiv \left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = 0. \tag{8.3}$$

---

[1] This chapter follows "A Multivector Derivative Approach to Lagrangian Field Theory," by A. Lasenby, C. Doran, and S. Gull, Feb. 9, 1993 available at http://www.mrao.cam.ac.uk/ cjld1/pages/publications.htm

Then

$$L\left(X_i',\dot{X}_i',t\right) = L\left(X_i+\epsilon Y_i,\dot{X}_i+\epsilon\dot{Y}_i,t\right) \tag{8.4}$$

$$= L\left(X_i,\dot{X}_i,t\right) + \epsilon\sum_{i=1}^{m}\left(Y_i * \partial_{X_i}L + \dot{Y}_i * \partial_{\dot{X}_i}L\right) \tag{8.5}$$

$$S = \int_{t_1}^{t_2} dt\left(L\left(X_i,\dot{X}_i,t\right) + \epsilon\sum_{i=1}^{m}\left(Y_i * \partial_{X_i}L + \dot{Y}_i * \partial_{\dot{X}_i}L\right)\right) \tag{8.6}$$

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int_{t_1}^{t_2} dt\sum_{i=1}^{m}\left(Y_i * \partial_{X_i}L + \dot{Y}_i * \partial_{\dot{X}_i}L\right) \tag{8.7}$$

$$= \int_{t_1}^{t_2} dt\sum_{i=1}^{m} Y_i * \left(\partial_{X_i}L - \frac{d}{dt}\left(\partial_{\dot{X}_i}L\right)\right) \tag{8.8}$$

where we use the definition of the multivector derivative to go from equation 8.4 to equation 8.5 and then use integration by parts with respect to time to go from equation 8.7 to equation 8.8. Since in equation 8.8 the Y_i's are arbitrary $\delta S = 0$ implies that the Lagrangian equations of motion are

$$\partial_{X_i}L - \frac{d}{dt}\left(\partial_{\dot{X}_i}L\right) = 0, \quad \forall i = 1,\ldots,m. \tag{8.9}$$

The multivector derivative insures that there are as many equations as there are grades present in the X_i, which implies there are the same number of equations as there are degrees of freedom in the system.

8.1.2 Symmetries and Conservation Laws

Consider a scalar parametrised transformation of the dynamical variables

$$X_i' = X_i'\left(X_i,\alpha\right), \tag{8.10}$$

where $X_i'\left(X_i,0\right) = X_i$. Now define

$$\delta X_i \equiv \left.\frac{dX_i'}{d\alpha}\right|_{\alpha=0}. \tag{8.11}$$

and a transformed Lagrangian

$$L'\left(X_i,\dot{X}_i,t\right) \equiv L\left(X_i',\dot{X}_i',t\right). \tag{8.12}$$

Then

$$\begin{aligned}
\left.\frac{dL'}{d\alpha}\right|_{\alpha=0} &= \sum_{i=1}^{m}\left(\delta X_i * \partial_{X'_i}L' + \delta\dot{X}_i * \partial_{\dot{X}'_i}L'\right) \\
&= \sum_{i=1}^{m}\left(\delta X_i * \partial_{X'_i}L' + \frac{d}{dt}\left(\delta X_i * \partial_{\dot{X}'_i}L'\right) - \delta X_i * \frac{d}{dt}\left(\partial_{\dot{X}'_i}L'\right)\right) \\
&= \sum_{i=1}^{m}\left(\delta X_i * \left(\partial_{X'_i}L' - \frac{d}{dt}\left(\partial_{\dot{X}'_i}L'\right)\right) + \frac{d}{dt}\left(\delta X_i * \partial_{\dot{X}'_i}L'\right)\right). \qquad (8.13)
\end{aligned}$$

If the X'_i's satisfy equation 8.9 equation 8.13 can be rewritten as

$$\left.\frac{dL'}{d\alpha}\right|_{\alpha=0} = \frac{d}{dt}\sum_{i=1}^{m}\left(\delta X_i * \partial_{\dot{X}_i}L\right). \qquad (8.14)$$

Noether's theorem is -

$$\left.\frac{dL'}{d\alpha}\right|_{\alpha=0} = 0 \implies \sum_{i=1}^{m}\left(\delta X_i * \partial_{\dot{X}_i}L\right) = \text{conserved quantity} \qquad (8.15)$$

From D& L -

"If the transformation is a symmetry of the Lagrangian, then L' is independent of α. In this case we immediately establish that a conjugate quantity is conserved. That is, symmetries of the Lagrangian produce conjugate conserved quantities. This is Noether's theorem, and it is valuable for extracting conserved quantities from dynamical systems. The fact that the derivation of equation 8.14 assumed the equations of motion were satisfied means that the quantity is conserved 'on-shell'. Some symmetries can also be extended 'off-shell', which becomes an important issue in quantum and super symmetric systems."

A more general treatment of symmetries and conservation is possible if we do not limit ourselves to a scalar parametrization. Instead let $X'_i = X'_i\left(X_i, M\right)$ where M is a multivector parameter. Then let

$$L' = L\left(X'_i, \dot{X}'_i, t\right) \qquad (8.16)$$

and calculate the multivector derivative of L' with respect to M using the chain rule (summation convention for repeated indices) first noting that

$$\frac{\partial}{\partial t}\left(\left(\left(A * \partial_M\right) X'_i\right) * \partial_{\dot{X}'_i}L'\right) = \left(\left(A * \partial_M\right)\dot{X}'_i\right) * \partial_{\dot{X}'_i}L' + \left(\left(A * \partial_M\right) X'_i\right) * \frac{\partial}{\partial t}\left(\partial_{\dot{X}'_i}L'\right) \qquad (8.17)$$

and then calculating

$$\begin{aligned}(A*\partial_M)\,L' &= ((A*\partial_M)\,X_i')*\partial_{X_i'}L' + \left((A*\partial_M)\,\dot{X}_i'\right)*\partial_{\dot{X}_i'}L' \\ &= ((A*\partial_M)\,X_i')*\left(\partial_{X_i'}L' - \frac{\partial}{\partial t}\left(\partial_{\dot{X}_i'}L'\right)\right) + \frac{\partial}{\partial t}\left(((A*\partial_M)\,X_i')*\partial_{\dot{X}_i'}L'\right). \quad (8.18)\end{aligned}$$

If we assume that the X_i''s satisfy the equations of motion we have

$$(A*\partial_M)\,L' = \frac{\partial}{\partial t}\left(((A*\partial_M)\,X_i')*\partial_{\dot{X}_i'}L'\right) \quad (8.19)$$

and differentiating equation 8.19 with respect to A (use equation 6.43) gives

$$\begin{aligned}\partial_M L' &= \frac{\partial}{\partial t}\left(\partial_A\left((A*\partial_M)\,X_i'\right)*\partial_{\dot{X}_i'}L'\right) \\ &= \frac{\partial}{\partial t}\left((\partial_M X_i')*\partial_{\dot{X}_i'}L'\right). \quad (8.20)\end{aligned}$$

Equation 8.20 is a generalization of Noether's theorem since if $\partial_M L' = 0$ then the parametrization M is a symmetry of the Lagrangian and $(\partial_M X_i')*\partial_{\dot{X}_i'}L'$ is a conserved quantity.

8.1.3 Examples of Lagrangian Symmetries

Time Translation

Consider the symmetry of time translation

$$X_i'(t,\alpha) = X_i(t+\alpha) \quad (8.21)$$

so that

$$\delta X_i = \left.\frac{dX_i'}{d\alpha}\right|_{\alpha=0} = \dot{X}_i, \quad (8.22)$$

and

$$\begin{aligned}\left.\frac{dL}{d\alpha}\right|_{\alpha=0} &= \frac{d}{dt}\sum_{i=1}^{m}\left(\dot{X}_i*\partial_{\dot{X}_i}L\right) \quad (8.23)\\ 0 &= \frac{d}{dt}\left(\sum_{i=1}^{m}\left(\dot{X}_i*\partial_{\dot{X}_i}L\right) - L\right). \quad (8.24)\end{aligned}$$

The conserved quantity is the Hamiltonian

$$H = \sum_{i=1}^{m} \left(\dot{X}_i * \partial_{\dot{X}_i} L \right) - L. \tag{8.25}$$

In terms of the generalized momenta

$$P_i = \partial_{\dot{X}_i} L, \tag{8.26}$$

so that

$$H = \sum_{i=1}^{m} \left(\dot{X}_i * P_i \right) - L. \tag{8.27}$$

Central Forces

Let the Lagrangian variables be x_i the vector position of the i^{th} particle in an ensemble of N particles with a Lagrangian of the form

$$L = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{x}_i^2 - \sum_{i=1}^{N} \sum_{j<i}^{N} V_{ij} \left(|x_i - x_j| \right) \tag{8.28}$$

which represent a classical system with central forces between each pair of particles.

First consider a translational invariance so that

$$x_i' = x_i + \alpha c \tag{8.29}$$

where α is a scalar parameter and c is a constant vector. Then

$$\delta x_i' = c \tag{8.30}$$

and

$$L' = L \tag{8.31}$$

so that the conserved quantity is (equation 8.15)

$$\begin{aligned} \sum_{i=1}^{N} \delta x_i * \partial_{\dot{x}_i} L &= c * \sum_{i=1}^{N} \partial_{\dot{x}_i} L \\ c \cdot p &= c \cdot \sum_{i=1}^{N} m_i \dot{x}_i \\ p &= \sum_{i=1}^{N} m_i \dot{x}_i \end{aligned} \tag{8.32}$$

since c is an arbitrary vector the vector p is also conserved and is the linear momentum of the system.

Now consider a rotational invariance where

$$x'_i = e^{\alpha B/2} x_i e^{-\alpha B/2} \tag{8.33}$$

where B is an arbitrary normalized ($B^2 = -1$) bivector in 3-dimensions and α is the scalar angle of rotation. Then again $L' = L$ since rotations leave $\dot{x}_i^2$ and $|x_i - x_j|$ unchanged and

$$\begin{aligned} \frac{dx'_i}{d\alpha} &= \frac{1}{2}\left(Be^{\alpha B/2} x_i e^{-\alpha B/2} - e^{\alpha B/2} x_i e^{-\alpha B/2} B\right) \\ \delta x'_i &= \frac{1}{2}\left(Bx_i - x_i B\right) \\ &= B \cdot x_i. \end{aligned} \tag{8.34}$$

Remember that since $B\cdot x_i$ is a vector and the scalar product ($*$) of two vectors is the dot product we have for a conserved quantity

$$\sum_{i=1}^{N} (B\cdot x_i)\cdot(\partial_{\dot{x}_i} L) = \sum_{i=1}^{N} m_i (B\cdot x_i)\cdot \dot{x}_i \tag{8.35}$$

$$= B\cdot \sum_{i=1}^{N} m_i (x_i \wedge \dot{x}_i) \tag{8.36}$$

$$= B\cdot J \tag{8.37}$$

$$J = \sum_{i=1}^{N} m_i (x_i \wedge \dot{x}_i) \tag{8.38}$$

where we go from equation 8.35 to equation 8.36 by using the identity in equation B.12. Then since equation 8.35 is conserved for any bivector B, the angular momentum bivector, J, of the system is conserved.

8.2 Lagrangian Theory for Continuous Systems

For ease of notation we define

$$A\overleftarrow{\nabla} \equiv \dot{A}\dot{\nabla}. \tag{8.39}$$

This is done for the situation that we are left differentiating a group of symbols. For example consider

$$(ABC)\overleftarrow{\nabla} = (A\dot{B}C)\dot{\nabla}. \tag{8.40}$$

The r.h.s. of equation 8.40 could be ambiguous in that could the overdot only apply to the B variable. Thus we will use the convention of equation 8.39 to denote differentiation of the group immediately to the left of the derivative. Another convention we could use to denote the same operation is

$$\widehat{ABC}\hat{\nabla} = (ABC)\overleftarrow{\nabla} \tag{8.41}$$

since using the "hat" symbol is unambiguous with respect to what symbols we are applying the differentiation operator to since the "hat" can extend over all the relevant symbols.

8.2.1 The Euler Lagrange Equations

Let $\psi_i\left(x\right)$ be a set of multivector fields and assume the Lagrangian density, $\mathcal{L}$, is a scalar function $\mathcal{L}\left(\psi_i, \nabla\psi_i, x\right)$ so that the action, S, of the continuous system is given by

$$S = \int_V \left|dx^n\right| \mathcal{L}\left(\psi_i, \nabla\psi_i, x\right), \tag{8.42}$$

where V is a compact n-dimensional volume. The equations of motion are given by minimizing S using the standard method of the calculus of variations where we define

$$\psi_i'\left(x\right) = \psi_i\left(x\right) + \epsilon\phi_i\left(x\right), \tag{8.43}$$

and assume that $\psi_i\left(x\right)$ yields an extrema of S and that $\phi_i\left(x\right) = 0$ for all $x \in \partial V$. Then to get an extrema we need to define

$$S\left(\epsilon\right) = \int_V \left|dx^n\right| \mathcal{L}\left(\psi_i + \epsilon\phi_i, \nabla\psi_i + \epsilon\nabla\phi_i, x\right), \tag{8.44}$$

so that $S\left(0\right)$ is an extrema if $\dfrac{\partial S}{\partial\epsilon} = 0$. Let us evaluate $\dfrac{\partial S}{\partial\epsilon}$ (summation convention)

$$\left.\frac{\partial S}{\partial\epsilon}\right|_{\epsilon=0} = \int_V \left|dx^n\right| \left(\left(\phi_i * \partial_{\psi_i}\right)\mathcal{L} + \left(\nabla\phi_i * \partial_{\nabla\psi_i}\right)\mathcal{L}\right). \tag{8.45}$$

Start by reducing the second term in the parenthesis on the r.h.s. of equation 8.45 using RR5 (Appendix C)

$$\begin{aligned}
\left(\nabla\phi_i * \partial_{\nabla\psi_i}\right)\mathcal{L} &= \left\langle\left(\nabla\phi_i\right)\partial_{\nabla\psi_i}\right\rangle\mathcal{L} \\
&= \left\langle\left(\nabla\phi_i\right)\partial_{\nabla\psi_i}\mathcal{L}\right\rangle \\
&= \left\langle\nabla\left(\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right) - \hat{\nabla}\phi_i\left(\widehat{\partial_{\nabla\psi_i}\mathcal{L}}\right)\right\rangle \\
&= \nabla\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1 - \left\langle\hat{\nabla}\phi_i\left(\widehat{\partial_{\nabla\psi_i}\mathcal{L}}\right)\right\rangle \\
&= \nabla\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1 - \left\langle\phi_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right\rangle \\
&= \nabla\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1 - \phi_i * \left(\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right). \qquad (8.46)
\end{aligned}$$

So that equation 8.45 becomes

$$\begin{aligned}
\left.\frac{\partial S}{\partial\epsilon}\right|_{\epsilon=0} &= \int_V \left|dx^n\right|\phi_i * \left(\partial_{\psi_i}\mathcal{L} - \left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right) + \int_V \left|dx^n\right|\nabla\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1 \\
&= \int_V \left|dx^n\right|\phi_i * \left(\partial_{\psi_i}\mathcal{L} - \left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right) + \int_{\partial V} \left|dS^{n-1}\right| n\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1 \\
&= \int_V \left|dx^n\right|\phi_i * \left(\partial_{\psi_i}\mathcal{L} - \left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right). \qquad (8.47)
\end{aligned}$$

The $\int_V \left|dx^n\right|\nabla\cdot\left\langle\phi_i\partial_{\nabla\psi_i}\mathcal{L}\right\rangle_1$ term is found to be zero by using the generalized divergence theorem (equation 4.84) and the fact that the ϕ_i's are zero on ∂V. If ϕ_i is a pure r-grade multivector we have by the properties of the scalar product the following Lagrangian field equations

$$\left\langle\partial_{\psi_i}\mathcal{L} - \left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla}\right\rangle_r = 0 \qquad (8.48)$$

or

$$\left\langle\partial_{\psi_i^\dagger}\mathcal{L} - \nabla\left(\partial_{(\nabla\psi_i)^\dagger}\mathcal{L}\right)\right\rangle_r = 0. \qquad (8.49)$$

For the more general case of a ϕ_i being a mixed grade multivector the Lagrangian field equations are

$$\partial_{\psi_i}\mathcal{L} - \left(\partial_{\nabla\psi_i}\mathcal{L}\right)\overleftarrow{\nabla} = 0 \qquad (8.50)$$

or

$$\partial_{\psi_i^\dagger}\mathcal{L} - \nabla\left(\partial_{(\nabla\psi_i)^\dagger}\mathcal{L}\right) = 0. \qquad (8.51)$$

Note that since equation 8.50 is true for ψ_i being any kind of multivector field we have derived the field equations for vectors, tensors (antisymmetric), spinors, or any combination thereof.

8.2.2 Symmetries and Conservation Laws

We proceed as in section 8.1.2 and let $\psi'_i = \psi'_i\left(\psi_i, M\right)$ where M is a multivector parameter. Then

$$\mathcal{L}'\left(\psi_i, \nabla\psi_i\right) \equiv \mathcal{L}\left(\psi'_i, \nabla\psi'_i\right) \tag{8.52}$$

and using equation 6.39 and RR5

$$\begin{aligned}
\left(A*\partial_M\right)\mathcal{L}' &= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\psi'_i}\mathcal{L}'\right) + \left(\left(A*\partial_M\right)\nabla\psi'_i\right)*\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\\
&= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\psi'_i}\mathcal{L}'\right) + \left\langle\left(\left(A*\partial_M\right)\nabla\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle\\
&= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\psi'_i}\mathcal{L}'\right) + \left\langle\nabla\left(\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right) - \hat{\nabla}\left(\left(\left(A*\partial_M\right)\psi'_i\right)\left(\widehat{\partial_{\nabla\psi'_i}\mathcal{L}'}\right)\right)\right\rangle\\
&= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\psi'_i}\mathcal{L}'\right) + \nabla\cdot\left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1 - \left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\overleftarrow{\nabla}\right\rangle\\
&= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\psi'_i}\mathcal{L}'\right) + \nabla\cdot\left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1 - \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\overleftarrow{\nabla}\\
&= \left(\left(A*\partial_M\right)\psi'_i\right)*\left(\left(\partial_{\psi'_i}\mathcal{L}'\right) - \left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\overleftarrow{\nabla}\right) + \nabla\cdot\left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1
\end{aligned} \tag{8.53}$$

and if the Euler-Lagrange equations are satisfied we have

$$\left(A*\partial_M\right)\mathcal{L}' = \nabla\cdot\left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1 \tag{8.54}$$

and by equation 6.43

$$\partial_M\mathcal{L}' = \partial_A\left(\nabla\cdot\left\langle\left(\left(A*\partial_M\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1\right). \tag{8.55}$$

If $\partial_M\mathcal{L}' = 0$, equation 8.55 is the most general form of Noether's theorem for the scalar valued multivector Lagrangian density.

If in equation 8.55 M is a scalar α and B is a scalar β we have

$$\begin{aligned}
\partial_\alpha\mathcal{L}' &= \partial_\beta\left(\nabla\cdot\left\langle\left(\left(\beta\partial_\alpha\right)\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1\right)\\
&= \nabla\cdot\left\langle\left(\partial_\alpha\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1.
\end{aligned} \tag{8.56}$$

Thus if $\alpha = 0$ in equation 8.56 we have

$$\left.\partial_\alpha\mathcal{L}'\right|_{\alpha=0} = \nabla\cdot\left.\left\langle\left(\partial_\alpha\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1\right|_{\alpha=0} \tag{8.57}$$

which corresponds to an differential transformation ($\partial_\alpha\mathcal{L}' = 0$ is a global transformation). If $\left.\partial_\alpha\mathcal{L}'\right|_{\alpha=0} = 0$ the conserved current is

$$j = \left.\left\langle\left(\partial_\alpha\psi'_i\right)\left(\partial_{\nabla\psi'_i}\mathcal{L}'\right)\right\rangle_1\right|_{\alpha=0} \tag{8.58}$$

with conservation law

$$\nabla \cdot j = 0. \tag{8.59}$$

If $\partial_\alpha \mathcal{L}'|_{\alpha=0} \neq 0$ we have by the chain rule that ($g(x) = \partial_\alpha x'|_{\alpha=0}$)

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \partial_\alpha x'|_{\alpha=0} \cdot \nabla \mathcal{L}'|_{\alpha=0} = g(x) \cdot \nabla \mathcal{L} \tag{8.60}$$

and consider

$$\nabla \cdot (g\mathcal{L}) = (\nabla \cdot g)\,\mathcal{L} + g \cdot \nabla \mathcal{L}. \tag{8.61}$$

If $\nabla \cdot g = 0$ we can write $\partial_\alpha \mathcal{L}'|_{\alpha=0}$ as a divergence so that

$$\nabla \cdot \big(\langle (\partial_\alpha \psi_i'|_{\alpha=0})\,(\partial_{\nabla\psi_i}\mathcal{L}) \rangle_1 - (\partial_\alpha x'|_{\alpha=0})\,\mathcal{L} \big) = 0 \tag{8.62}$$

and

$$j = \langle (\partial_\alpha \psi_i'|_{\alpha=0})\,(\partial_{\nabla\psi_i}\mathcal{L}) \rangle_1 - (\partial_\alpha x'|_{\alpha=0})\,\mathcal{L} \tag{8.63}$$

is a conserved current if $\nabla \cdot \partial_\alpha x'|_{\alpha=0} = 0$.

Note that since $\partial_\alpha x'|_{\alpha=0}$ is the derivative of a vector with respect to a scalar it itself is a vector. Thus the conserved j is always a vector that could be a linear function of a vector, bivector, etc. depending upon the type of transformation (vector for affine and bivector for rotation). However, the conserved quantity of interest may be other than a vector such as the stress-energy tensor or the angular momentum bivector. In these cases the conserved vector current must be transformed to the conserved quantity of interest via the general adjoint transformation $A * \overline{j}(B) = B * \underline{j}(A)$.

8.2.3 Space-Time Transformations and their Conjugate Tensors

The canonical stress-energy tensor is the current associated with the symmetries of space-time translations. As a function of the parameter α we have

$$x' = x + \alpha n \tag{8.64}$$

$$\psi_i'(x) = \psi_i(x + \alpha n) \tag{8.65}$$

Then

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \partial_\alpha \mathcal{L}(x + \alpha n)|_{\alpha=0} = n \cdot \nabla \mathcal{L} = \nabla \cdot (n\mathcal{L}) \tag{8.66}$$

$$\partial_\alpha \psi_i'|_{\alpha=0} = n \cdot \nabla \psi_i \tag{8.67}$$

and equation 8.56 becomes

$$\nabla \cdot (n\mathcal{L}) = \nabla \cdot \langle (n \cdot \nabla \psi_i)\,(\partial_{\nabla\psi_i}\mathcal{L}) \rangle_1 \tag{8.68}$$

so that

$$\nabla\cdot\overline{T}(n)\equiv\nabla\cdot\left\langle(n\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)-n\mathcal{L}\right\rangle_1=0. \tag{8.69}$$

Thus the conserved current, $\overline{T}(n)$, is a linear vector function of a vector n, a tensor of rank 2. In order put the stress-energy tensor into the standard form we need the adjoint, $\underline{T}(n)$, of $\overline{T}(n)$ (we are using that the adjoint of the adjoint is the original linear transformation).

$$\begin{aligned}\overline{T}(n)&=\left\langle(n\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1-n\mathcal{L} &(8.70)\\&=\left(n\cdot\dot{\nabla}\right)\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1-n\mathcal{L}. &(8.71)\end{aligned}$$

Using equation 6.34 we get

$$\begin{aligned}\underline{T}(n)&=\partial_m\left\langle\left(m\cdot\dot{\nabla}\right)\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 n\right\rangle-n\mathcal{L}\\&=\partial_m\left(m\cdot\dot{\nabla}\right)\left\langle\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 n\right\rangle-n\mathcal{L}\\&=\dot{\nabla}\left\langle\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 n\right\rangle-n\mathcal{L}\\&=\dot{\nabla}\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)n\right\rangle-n\mathcal{L}. \qquad (8.72)\end{aligned}$$

From equation 8.69 it follows that

$$\begin{aligned}0=\nabla\cdot\overline{T}(n)&=n\cdot\dot{T}\left(\dot{\nabla}\right), &(8.73)\\ \dot{T}\left(\dot{\nabla}\right)&=0, &(8.74)\end{aligned}$$

or in rectangular coordinates

$$\dot{T}\left(\dot{\nabla}\right)=\dot{T}\left(e^{\mu}\frac{\dot{\partial}}{\partial x^{\mu}}\right)=\frac{\partial}{\partial x^{\mu}}T\left(e^{\mu}\right)=\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}e_{\nu}. \tag{8.75}$$

Thus in standard tensor notation

$$\frac{\partial T^{\mu\nu}}{\partial x^{\mu}}=0, \tag{8.76}$$

so that $T(n)$ is a conserved tensor.

Now consider rotational transformations and for now assume the ψ_i transform as vectors so that

$$\begin{aligned}x'&=e^{-\frac{\alpha B}{2}}xe^{\frac{\alpha B}{2}} &(8.77)\\ \psi_i'(x)&=e^{\frac{\alpha B}{2}}\psi_i\left(x'\right)e^{-\frac{\alpha B}{2}}, &(8.78)\end{aligned}$$

Note that we are considering *active* transformations. The transformation of x to x' maps one distinct point in space-time to another distinct point in space-time. We are not considering *passive* transformations which are only a transformation of coordinates and the position vector does not move in space-time. Because the transformation is *active* the sense of rotation of the vector field $\psi(x)$ is opposite that of the rotation of the space-time position vector[2]. Thus

$$\begin{aligned}
\partial_\alpha x' &= \frac{1}{2}e^{-\frac{\alpha B}{2}}\left(xB - Bx\right)e^{\frac{\alpha B}{2}} \\
\left.\partial_\alpha x'\right|_{\alpha=0} &= x\cdot B \qquad (8.79) \\
\partial_\alpha \psi_i' &= \frac{1}{2}e^{\frac{\alpha B}{2}}\left(B\psi_i\left(x'\right) - \psi_i\left(x'\right)B\right)e^{-\frac{\alpha B}{2}} + e^{\frac{\alpha B}{2}}\left(\partial_\alpha \psi_i\left(x'\right)\right)e^{-\frac{\alpha B}{2}} \\
&= e^{\frac{\alpha B}{2}}\left(B\times\psi_i\left(x'\right)\right)e^{-\frac{\alpha B}{2}} + e^{\frac{\alpha B}{2}}\left(\partial_\alpha x'\cdot\nabla\psi_i\left(x'\right)\right)e^{-\frac{\alpha B}{2}} \qquad (8.80) \\
\left.\partial_\alpha \psi_i'\right|_{\alpha=0} &= B\times\psi_i\left(x\right) + \left.\partial_\alpha x'\right|_{\alpha=0}\cdot\nabla\psi_i\left(x\right) \\
&= B\times\psi_i + \left(x\cdot B\right)\cdot\nabla\psi_i = \psi_i\cdot B + \left(x\cdot B\right)\cdot\nabla\psi_i. \qquad (8.81)
\end{aligned}$$

Thus

$$\nabla\cdot\left(\left.\partial_\alpha x'\right|_{\alpha=0}\right) = \nabla\cdot\left(x\cdot B\right) = -\left(B\cdot\dot{x}\right)\cdot\dot{\nabla} = -B\cdot\left(\dot{x}\wedge\dot{\nabla}\right) = B\cdot\left(\nabla\wedge x\right) = 0 \qquad (8.82)$$

since $\nabla\wedge x = 0$[3] the derivative of the transformed Lagrangian at $\alpha = 0$ is a pure divergence,

$$\left.\partial_\alpha \mathcal{L}'\right|_{\alpha=0} = \nabla\cdot\left(\left(x\cdot B\right)\mathcal{L}\right). \qquad (8.83)$$

$$\nabla\cdot\left.\left\langle\left(\partial_\alpha\psi_i'\right)\left(\partial_{\nabla\psi_i'}\mathcal{L}\right)\right\rangle_1\right|_{\alpha=0} = \nabla\cdot\left\langle\left(B\times\psi_i - \left(B\cdot x\right)\cdot\nabla\psi_i\right)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 \qquad (8.84)$$

---

[2] Consider an observer at location x that is rotated to location x'. If he is rotated in a clockwise sense he observes the vector field to be rotating in a counter clockwise sense.

[3] Consider

$$\begin{aligned}
\nabla\wedge x &= e^\nu\frac{\partial}{\partial x^\nu}\wedge x_\eta e^\eta \\
&= \frac{\partial}{\partial x^\nu}x_\eta e^\nu\wedge e^\eta \\
&= \frac{\partial}{\partial x^\nu}g_{\eta\mu}x^\mu e^\nu\wedge e^\eta \\
&= \frac{\partial x^\mu}{\partial x^\nu}g_{\eta\mu}e^\nu\wedge e^\eta \\
&= \delta^\mu_\nu g_{\eta\mu}e^\nu\wedge e^\eta \\
&= g_{\eta\nu}e^\nu\wedge e^\eta = 0
\end{aligned}$$

since $g_{\eta\nu}$ is symmetric and $e^\nu\wedge e^\eta$ is antisymmetric.

$$\begin{aligned}\nabla\cdot\overline{J}(B) &\equiv \nabla\cdot\left\langle(\partial_\alpha\psi_i')\left(\partial_{\nabla\psi_i'}\mathcal{L}\right)\right\rangle_1\Big|_{\alpha=0} - \partial_\alpha\mathcal{L}'\big|_{\alpha=0}\\ &= \nabla\cdot\Big(\left\langle(B\times\psi_i-(B\cdot x)\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1-(x\cdot B)\,\mathcal{L}\Big)\end{aligned} \tag{8.85}$$

$$\overline{J}(B) = \left\langle(B\times\psi_i-(B\cdot x)\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1-(x\cdot B)\,\mathcal{L}. \tag{8.86}$$

By Noether's theorem $\dot{\nabla}\cdot\dot{\overline{J}}(B)=0$ where $\overline{J}(B)$ is a conserved vector. So that

$$\dot{\nabla}\cdot\dot{\overline{J}}(B)=0 \quad\Rightarrow\quad \dot{\underline{J}}\left(\dot{\nabla}\right)\cdot B = 0 \text{ for all } B \quad\Rightarrow\quad \dot{\underline{J}}\left(\dot{\nabla}\right)=0$$

The adjoint functon $\underline{J}(n)$ is, therefore a conserved bivector-valued function of position (we can use $\cdot$ instead of $*$ since all the grades match up correctly).

Now consider the coordinate (tensor) representation of $\dot{\underline{J}}\left(\dot{\nabla}\right)=0$

$$n = n_\gamma \boldsymbol{e}^\gamma \tag{8.87}$$

$$\underline{J}(n) = J^{\mu\nu\gamma} n^\gamma \boldsymbol{e}_\mu\wedge\boldsymbol{e}_\nu \tag{8.88}$$

$$\dot{\underline{J}}\left(\dot{\nabla}\right) = \frac{\partial J^{\mu\nu\gamma}}{\partial x^\gamma}\boldsymbol{e}_\mu\wedge\boldsymbol{e}_\nu = 0 \tag{8.89}$$

$$\frac{\partial J^{\mu\nu\gamma}}{\partial x^\gamma} = 0 \tag{8.90}$$

To compute $\underline{J}(n)$ note that A is a bivector using equation 6.34[4] to calculate the adjoint of the adjoint and $(A*\partial_B)\,B = A$[5] and RR7[6] we get

$$\begin{aligned} A*\underline{J}(n) &= (A*\partial_B)\left\langle\overline{J}(B)\,n\right\rangle\\ &= (A*\partial_B)\left\langle\left\langle(B\times\psi_i-(B\cdot x)\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 n-(x\cdot B)\,\mathcal{L}n\right\rangle\\ &= \left\langle(A\times\psi_i-(A\cdot x)\cdot\nabla\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)-(x\cdot A)\,\mathcal{L}n\right\rangle\\ &= \left\langle(A\times\psi_i-(A\cdot(x\wedge\nabla))\,\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)-(x\cdot A)\,\mathcal{L}n\right\rangle.\end{aligned} \tag{8.91}$$

Using RR5[7] (Reduction Rule 5 in Appendix C) the first term on the r.h.s. of equation 8.91

---

[4] $\overline{F}(B) = \partial_A\left\langle\underline{F}(A)\,B\right\rangle$

[5] $(A*\partial_B)\,B = \lim\limits_{h\to 0}\dfrac{B+hA-B}{h} = A$

[6] $(\langle A\rangle_2\cdot a)\cdot b = \langle A\rangle_2\cdot(a\wedge b)$

[7] $\langle A_1\ldots A_k\rangle = \langle A_k A_1\ldots A_{k-1}\rangle$

reduces to

$$\begin{aligned}
\left\langle (A\times\psi_i)\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle &= \frac{1}{2}\left\langle (A\psi_i-\psi_iA)\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle \\
&= \frac{1}{2}\left\langle A\left\{\psi_i\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)-\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\psi_i\right\}\right\rangle \\
&= \left\langle A\left(\psi_i\times\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right)\right\rangle \\
&= A*\left\langle \psi_i\times\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2, \qquad (8.92)
\end{aligned}$$

using RR7 the second term reduces to

$$\begin{aligned}
\left\langle (A\cdot(x\wedge\nabla))\,\psi_i\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle &= \left(A\cdot\left(x\wedge\dot{\nabla}\right)\right)\left\langle \dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\right\rangle \\
&= A*\left(\left(x\wedge\dot{\nabla}\right)\left\langle \dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\right\rangle\right), \qquad (8.93)
\end{aligned}$$

using RR5 again the third term reduces to

$$\begin{aligned}
\left\langle (x\cdot A)\,\mathcal{L}n\right\rangle &= \frac{1}{2}\left\langle (xA-Ax)\,\mathcal{L}n\right\rangle \\
&= \frac{1}{2}\left\langle A\,(nx-xn)\,\mathcal{L}\right\rangle \\
&= \left\langle A\,(n\wedge x)\,\mathcal{L}\right\rangle \\
&= A*(n\wedge x)\,\mathcal{L}. \qquad (8.94)
\end{aligned}$$

Combining equations 8.92, 8.93, and 8.94 reduces equation 8.91 to

$$\underline{J}(n) = \left\langle \psi_i\times\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2 - \left(x\wedge\dot{\nabla}\right)\left\langle \dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\right\rangle - (n\wedge x)\,\mathcal{L} \qquad (8.95)$$

where $\underline{J}(n)$ is the angular momentum bivector for the vector field. Using equation 8.93 we can reduce equation 8.95 to

$$\underline{J}(n) = T(n)\wedge x + \left\langle \psi_i\times\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2. \qquad (8.96)$$

If ψ_i is a spinor instead of a vector the transformation law is

$$\psi_i'(x) = e^{\frac{\alpha B}{2}}\psi_i(x') \qquad (8.97)$$

so that

$$\partial_\alpha\psi_i' = e^{\frac{\alpha B}{2}}\frac{B}{2}\psi_i(x') + e^{\frac{\alpha B}{2}}\partial_\alpha x'\psi_i(x') \qquad (8.98)$$

and

$$\partial_\alpha\psi_i'\big|_{\alpha=0} = \frac{B}{2}\psi_i + (x\cdot B)\cdot\nabla\psi_i. \qquad (8.99)$$

Equations 8.86 and 8.95 become

$$\overline{J}\left(B\right) = \left\langle \left(\frac{B}{2}\psi_i - \left(B\cdot x\right)\cdot\nabla\psi_i\right)\left(\partial_{\nabla\psi_i}\mathcal{L}\right)\right\rangle_1 - \left(x\cdot B\right)\mathcal{L} \tag{8.100}$$

and

$$\underline{J}\left(n\right) = \left\langle \frac{\psi_i}{2}\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2 - \left(x\wedge\dot{\nabla}\right)\left\langle\dot{\psi}_i\partial_{\nabla\psi_i}\mathcal{L}n\right\rangle - \left(n\wedge x\right)\mathcal{L} \tag{8.101}$$

where $\underline{J}\left(n\right)$ is the angular momentum bivector for the spinor field.

Since spinors transforms the same as vectors under translations the expression (equation 8.72) for the stress-energy tensor of a spinor field is the same as for a vector or scalar field, $\underline{T}\left(n\right) = \dot{\nabla}\left\langle\dot{\psi}_i\left(\partial_{\nabla\psi_i}\mathcal{L}\right)n\right\rangle - n\mathcal{L}$, so that

$$\text{Spinor Field: } \underline{J}\left(n\right) = \underline{T}\left(n\right)\wedge x + \left\langle\frac{\psi_i}{2}\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2, \tag{8.102}$$

compared to

$$\text{Vector Field: } \underline{J}\left(n\right) = T\left(n\right)\wedge x + \left\langle\psi_i\times\left(\partial_{\nabla\psi_i}\mathcal{L}n\right)\right\rangle_2. \tag{8.103}$$

8.2.4 Case 1 - The Electromagnetic Field

Example of Lagrangian densities are the electromanetic field and the spinor field for an electron. For the electromagnetic field we have

$$\mathcal{L} = \frac{1}{2}F\cdot F - A\cdot J \tag{8.104}$$

where F $= \nabla\wedge A$ and using eq 6.15 and eq 6.17

$$\begin{aligned}
F\cdot F &= \left\langle\left(\nabla\wedge A\right)\left(\nabla\wedge A\right)\right\rangle \\
&= \left\langle\frac{1}{2}\left(\nabla A - \left(\nabla A\right)^\dagger\right)\frac{1}{2}\left(\nabla A - \left(\nabla A\right)^\dagger\right)\right\rangle \\
&= \frac{1}{4}\left\langle\left(\nabla A - \left(\nabla A\right)^\dagger\right)^2\right\rangle \\
&= \frac{1}{4}\left\langle\nabla A\nabla A - \nabla A\left(\nabla A\right)^\dagger - \left(\nabla A\right)^\dagger\nabla A + \left(\nabla A\right)^\dagger\left(\nabla A\right)^\dagger\right\rangle \\
&= \frac{1}{4}\left(\left(\nabla A\right)*\left(\nabla A\right) - \left(\nabla A\right)*\left(\nabla A\right)^\dagger - \left(\nabla A\right)^\dagger*\left(\nabla A\right) + \left(\nabla A\right)^\dagger*\left(\nabla A\right)^\dagger\right) \\
&= \frac{1}{2}\left(\left(\nabla A\right)*\left(\nabla A\right) - \left(\nabla A\right)*\left(\nabla A\right)^\dagger\right)
\end{aligned} \tag{8.105}$$

Now calculate using eq 6.47 and eq 6.51

$$\begin{aligned}\partial_{(\nabla A)^\dagger} F^2 &= \frac{1}{2}\partial_{(\nabla A)^\dagger}\left((\nabla A)*(\nabla A)-(\nabla A)*(\nabla A)^\dagger\right)\\ &= (\nabla A)^\dagger - \nabla A\\ &= -2\nabla\wedge A \qquad (8.106)\end{aligned}$$

We also have by eq 6.44 and that $A^\dagger = A$

$$\begin{aligned}\partial_{A^\dagger}(A\cdot J) &= \partial_{A^\dagger}(J*A)\\ &= J \qquad (8.107)\end{aligned}$$

so that

$$\partial_{A^\dagger}\mathcal{L} - \nabla\left(\partial_{(\nabla A)^\dagger}\mathcal{L}\right) = J - \nabla(\nabla\wedge A) = 0 \qquad (8.108)$$

$$\nabla F = J \qquad (8.109)$$

8.2.5 Case 2 - The Dirac Field

For the Dirac field

$$\begin{aligned}\mathcal{L} &= \left\langle \nabla\psi I\gamma_z\psi^\dagger - eA\psi\gamma_t\psi^\dagger - m\psi\psi^\dagger\right\rangle\\ &= (\nabla\psi)*\left(I\gamma_z\psi^\dagger\right) - \left\langle eA\psi\gamma_t\psi^\dagger\right\rangle - m\psi*\psi^\dagger \qquad (8.110)\\ &= (\nabla\psi I\gamma_z)*\psi^\dagger - \left\langle eA\psi\gamma_t\psi^\dagger\right\rangle - m\psi*\psi^\dagger \qquad (8.111)\end{aligned}$$

The only term in eq 8.110 and eq 8.111 that we cannot immediately differentiate is $\partial_{\psi^\dagger}\left\langle eA\psi\gamma_t\psi^\dagger\right\rangle$. To perform this operation let $\psi^\dagger = X$ and use the definition of the scalar directional derivative

$$\begin{aligned}(B*\partial_X)\left\langle eAX^\dagger\gamma_t X\right\rangle &= \lim_{h\to 0}\frac{\left\langle eA\left(X^\dagger + hB^\dagger\right)\gamma_t(X+hB) - eAX^\dagger\gamma_t X\right\rangle}{h}\\ &= \left\langle eAB^\dagger\gamma_t X + eAX^\dagger\gamma_t B\right\rangle\\ &= B^\dagger*(e\gamma_t XA) + B*\left(eAX^\dagger\gamma_t\right)\\ &= B*\left(eAX^\dagger\gamma_t\right) + B*\left(eAX^\dagger\gamma_t\right)\\ &= B*\left(2eAX^\dagger\gamma_t\right) \qquad (8.112)\\ \partial_X\left\langle eAX^\dagger\gamma_t X\right\rangle &= 2eAX^\dagger\gamma_t \qquad (8.113)\\ \partial_{\psi^\dagger}\left\langle eA\psi\gamma_t\psi^\dagger\right\rangle &= 2eA\psi\gamma_t \qquad (8.114)\end{aligned}$$

The other multivector derivatives are evaluated using the formulas in section 6.2

$$\partial_{(\nabla\psi)^\dagger}\left((\nabla\psi)*\left(I\gamma_z\psi^\dagger\right)\right)=\psi\gamma_zI^\dagger=\psi\gamma_zI=-\psi I\gamma_z \tag{8.115}$$

$$\partial_{\psi^\dagger}\left((\nabla\psi I\gamma_z)*\psi^\dagger\right)=\nabla\psi I\gamma_z \tag{8.116}$$

$$\partial_{\psi^\dagger}\left(m\psi*\psi^\dagger\right)=2m\psi. \tag{8.117}$$

The Lagrangian field equation are then

$$\begin{aligned}\partial_{\psi^\dagger}\mathcal{L}-\nabla\left(\partial_{(\nabla\psi)^\dagger}\mathcal{L}\right)&=\nabla\psi I\gamma_z-2eA\psi\gamma_t-2m\psi+\nabla\psi I\gamma_z=0\\&=2\left(\nabla\psi I\gamma_z-eA\psi\gamma_t-m\psi\right)=0\\0&=\nabla\psi I\gamma_z-eA\psi\gamma_t-m\psi.\end{aligned} \tag{8.118}$$

8.2.6 Case 3 - The Coupled Electromagnetic and Dirac Fields

For coupled electromagnetic and electron (Dirac) fields the coupled Lagrangian is

$$\begin{aligned}\mathcal{L}&=\left\langle\nabla\psi I\gamma_z\psi^\dagger-eA\psi\gamma_t\psi^\dagger-m\psi\psi^\dagger+\frac{1}{2}F^2\right\rangle\\&=(\nabla\psi)*\left(I\gamma_z\psi^\dagger\right)-\left\langle eA\psi\gamma_t\psi^\dagger\right\rangle-m\psi*\psi^\dagger+\frac{1}{2}F^2 &(8.119)\\&=(\nabla\psi I\gamma_z)*\psi^\dagger-eA*\left(\psi\gamma_t\psi^\dagger\right)-m\psi*\psi^\dagger+\frac{1}{2}F^2, &(8.120)\end{aligned}$$

where A is the 4-vector potential dynamical field and ψ is the spinor dynamical field. Between the previously calculated multivector derivatives for the electromagnetic and Dirac fields the only new derivative that needs to be calculated for the Lagrangian field equations is

$$\partial_{A^\dagger}\left(eA*\left(\psi\gamma_t\psi^\dagger\right)\right)=e\left(\psi\gamma_t\psi^\dagger\right)^\dagger=e\psi\gamma_t\psi^\dagger. \tag{8.121}$$

The Lagrangian equations for the coupled fields are then

$$\partial_{\psi^\dagger}\mathcal{L}-\nabla\left(\partial_{(\nabla\psi)^\dagger}\mathcal{L}\right)=2\left(\nabla\psi I\gamma_z-eA\psi\gamma_t-m\psi\right)=0 \tag{8.122}$$

$$\partial_{A^\dagger}\mathcal{L}-\nabla\left(\partial_{(\nabla A)^\dagger}\mathcal{L}\right)=-e\psi\gamma_t\psi^\dagger+\nabla\left(\nabla\wedge A\right)=0 \tag{8.123}$$

or

$$\nabla\psi I\gamma_z-eA\psi\gamma_t=m\psi \tag{8.124}$$

$$\nabla F=e\psi\gamma_t\psi^\dagger. \tag{8.125}$$

Chapter 9

Lie Groups as Spin Groups[1]

9.1 Introduction

A Lie group, G, is a group that is also a differentiable manifold. So that if $g \in G$ and $x \in \Re^N$ then there is a differentiable mapping $\phi : \Re^N \rightarrow G$ so that for each $g \in G$ we can define a tangent space $\mathcal{T}_g$.

A example (the one we are most concerned with) of a Lie group is the group of $n \times n$ non-singular matrices. The coordinates of the manifold are simply the elements of the matrix so that the dimension of the group manifold is n^2 and the matrix is obviously a continuous differentiable function of its coordinates.

A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear operator $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ called the *Lie bracket* which satifies ($x, y, z \in \mathfrak{g}$ and $\alpha, \beta \in \Re$):

$$[ax + by, z] = a\,[x, z] + b\,[y, z]\,, \tag{9.1}$$

$$[x, x] = 0, \tag{9.2}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0. \tag{9.3}$$

Equations (9.1) and (9.2) imply $[x, y] = -[y, x]$, while eq (9.3) is the Jacobi identity.

The purpose of the following analysis is to show that the Lie algebra of the general linear group of dimension n over the real numbers (the group of $n \times n$ invertible matrices), $GL(n, \Re)$, can be represented by a rotation group in an appropriate vector space, $\mathcal{M}$ (let (p, q) be the signature of the new vector space). Furthermore, since the rotation group, $SO\left(p, q\right)$ can be represented by the spin group, $Spin\left(p, q\right)$, in the same vector space. Then by use of geometric algebra we can construct any rotor, $R \in Spin\left(p, q\right)$, as $R = \mathrm{e}^{\frac{B}{2}}$ where B is a bivector in the geometric algebra of $\mathcal{M}$ and the bivectors form a Lie algebra under the commutator product.[2]

---

[1] This chapter follows [5]

The main trick in doing this is to construct the appropriate vector space, $\mathcal{M}$, with subspaces isomorphic to $\Re^n$ so that a rotation in $\mathcal{M}$ is equivalent to a general linear transformation in a subspace of $\mathcal{M}$ isomorphic to $\Re^n$. We might suspect that $\mathcal{M}$ cannot be a Euclidean space since a general linear transformation can cause the input vector to grow as well as shrink.

9.2 Simple Examples

9.2.1 $SO\left(2\right)$ - Special Orthogonal Group of Order 2

The group is represented by all 2×2 real matrices[3] $\underline{R}$ where[4] $\underline{R}\underline{R}^T = \underline{I}$ and $\det\left(\underline{R}\right) = 1$. The group product is matrix multiplication and it is a group since if $\underline{R}_1\underline{R}_1^T = \underline{I}$ and $\underline{R}_2\underline{R}_2^T = \underline{I}$ then

$$\left(\underline{R}_1\underline{R}_2\right)\left(\underline{R}_1\underline{R}_2\right)^T = \underline{R}_1\underline{R}_2\underline{R}_2^T\underline{R}_1^T = \underline{R}_1\underline{I}\underline{R}_1^T = \underline{R}_1\underline{R}_1^T = \underline{I} \tag{9.4}$$

$$\det\left(\underline{R}_1\underline{R}_2\right) = \det\left(\underline{R}_1\right)\det\left(\underline{R}_2\right) = 1. \tag{9.5}$$

$SO\left(2\right)$ is also a Lie group since all $\underline{R} \in SO\left(2\right)$ can be represented as a continuous function of the coordinate θ:

$$\underline{R}\left(\theta\right) = \left[\begin{array}{cc} \cos\left(\theta\right) & -\sin\left(\theta\right) \\ \sin\left(\theta\right) & \cos\left(\theta\right) \end{array}\right]. \tag{9.6}$$

In this case the spin representation of $SO\left(2\right)$ is trivial, namely[5] $R\left(\theta\right) = \mathrm{e}^{\frac{I}{2}\theta} = \cos\left(\frac{\theta}{2}\right) + I\sin\left(\frac{\theta}{2}\right)$, $R\left(\theta\right)^\dagger = \mathrm{e}^{-\frac{I}{2}\theta} = \cos\left(\frac{\theta}{2}\right) - I\sin\left(\frac{\theta}{2}\right)$ where I is the pseudo-scalar for $\mathcal{G}\left(\Re^2\right)$. Then we have $\underline{R}\left(\theta\right)a = R\left(\theta\right)aR\left(\theta\right)^\dagger$.

---

[2] Since we could be dealing with vector spaces of arbitrary signature (p, q) we use the following nomenclature:

$SO\left(p, q\right)$	Special orthogonal group in vector space with signature (p, q)
$SO\left(n\right)$	Special orthogonal group in vector space with signature $(n, 0)$
$Spin\left(p, q\right)$	Spin group in vector space with signature (p, q)
$Spin\left(n\right)$	Spin group in vector space with signature $(n, 0)$
$GL\left(p, q, \Re\right)$	General linear group in real vector space with signature (p, q)
$GL\left(n, \Re\right)$	General linear group in real vector space with signature $(n, 0)$

[3] We denote linear transformations with an underbar.

[4] In this case $\underline{I}$ is the identity matrix and not the pseudo scalar.

[5] For multivectors such as the rotor R there is no underbar. We have $\underline{R}\left(a\right) = \underline{R}a = RaR^\dagger$ where a is a vector.

9.2.2 $GL\left(2,\Re\right)$ - General Real Linear Group of Order 2

The group is represented by all 2×2 real matrices $\underline{A}$ where $\det\left(\underline{A}\right)\neq 0$. Again the the group product is matrix multiplication and it is a group because if $\underline{A}$ and $\underline{B}$ are 2×2 real matrices then $\det\left(\underline{AB}\right)=\det\left(\underline{A}\right)\det\left(\underline{B}\right)$ and if $\det\left(\underline{A}\right)\neq 0$ and $\det\left(\underline{B}\right)\neq 0$ then $\det\left(\underline{AB}\right)\neq 0$. Any member of the group is represented by the matrix ($a=(a^1,a^2,a^3,a^4)\in\Re^4$):

$$\underline{A}\left(a\right)=\left[\begin{array}{cc} a^1 & a^2 \\ a^3 & a^4 \end{array}\right]. \tag{9.7}$$

Thus any element $\underline{A}\in GL\left(2,\Re\right)$ is a continuous function of $a=(a^1,a^2,a^3,a^4)$. Thus $GL\left(2,\Re\right)$ is a four dimensional Lie group while $SO\left(2\right)$ is a one dimensional Lie group.

Another difference is that $SO\left(2\right)$ is compact while $GL\left(2,\Re\right)$ is not. $SO\left(2\right)$ is compact since for any convergent sequence (θ_i),

$$\lim_{i\rightarrow\infty}\underline{R}\left(\theta_i\right)\in SO\left(2\right).$$

$GL\left(2,\Re\right)$ is not compact since there is at least one convergent sequence (a_i) such that

$$\lim_{i\rightarrow\infty}\det\left(\underline{A}\left(a_i\right)\right)=0.$$

After we develop the required theory we will calculate the spin representation of $GL\left(2,\Re\right)$.

9.3 Properties of the Spin Group

The spin group, $Spin\left(p,q\right)$, of signature (p,q) is the group of even multivectors, R, such that $RR^\dagger=1$. Then if $R\in Spin\left(p,q\right)$ the rotation operator, $\underline{R}$, corresponding to R is defined by

$$\underline{R}x\equiv RxR^\dagger\ \ \forall\, x\in\Re^{p,q}. \tag{9.8}$$

9.3.1 Every Rotor is the Exponential of a Bivector

Now we wish to show that any rotor, R, can be written as $R=\mathrm{e}^B$ where B is a bivector.

To begin we must establish some properties of the rotor $R_a=\mathrm{e}^{B_a}$ where B_a is a 2-blade[6], which is to say that B_a defines a plane a by the relation $x\wedge B_a=0,\ x\in\Re^{p,q}$. Since B_a is a blade we have $B_a^2 \in \Re$. We evaluate R_a as follows:

---

[6] Remember that an n-blade is the wedge (outer) product of n vectors and defines and n-dimensional subspace.

$$
\begin{aligned}
&B_a^2 < 0 : \\
&\qquad R_a = \cos\left(\sqrt{-B_a^2}\right) + \sin\left(\sqrt{-B_a^2}\right)\frac{B_a}{\sqrt{-B_a^2}} \qquad (9.9)\\
&B_a^2 > 0 : \\
&\qquad R_a = \cosh\left(\sqrt{B_a^2}\right) + \sinh\left(\sqrt{B_a^2}\right)\frac{B_a}{\sqrt{B_a^2}}. \qquad (9.10)
\end{aligned}
$$

Likewise, if we have an expression of the form:

$$R_a = S_a + B_a, \tag{9.11}$$

where S_a is a scalar and B_a a bivector blade we can put it in the form of eq (9.9) or eq (9.10) if $S_a^2 - B_a^2 = 1$. This implies that $R = \mathrm{e}^{\theta_a \hat{B}_a}$, where $\hat{B}_a^2 = \pm 1$ is a normalized 2-blade and $\theta_a = \mathrm{Sin}^{-1}\left(\sqrt{-B_a^2}\right)$ or $\theta_a = \mathrm{Sinh}^{-1}\left(\sqrt{B_a^2}\right)$ depending on the sign of B_a^2.

Now consider the compound rotor $R = R_a R_b = \mathrm{e}^{B_a}\,\mathrm{e}^{B_b}$ where both B_a and B_b are 2-blades. If B_a and B_b commute (the planes defined by B_a and B_b do not intersect) then

$$R = R_a R_b = \mathrm{e}^{B_a}\,\mathrm{e}^{B_b} = \mathrm{e}^{B_a + B_b} = \mathrm{e}^{B_b}\,\mathrm{e}^{B_a} = R_b R_a, \tag{9.12}$$

and the rotors commute. If B_a and B_b do not commute the planes defined by B_a and B_b must intersect on a line as shown in figure (9.1).

Define the basis vectors for intersecting planes as shown in figure (9.1) by

$\boldsymbol{e}_a$	A unit vector in the plane B_a and is normal to $\boldsymbol{e}_m$.
$\boldsymbol{e}_m$	A unit vector along the intersection of a and b.
$\boldsymbol{e}_b$	A unit vector in the B_b and is normal to $\boldsymbol{e}_m$.

where unit vector means that the square of the basis vector can be ±1. The metric tensor for vector space spanned by these basis vectors is

$$g = \left(\begin{array}{ccc} \boldsymbol{e}_a^2 & 0 & \alpha \\ 0 & \boldsymbol{e}_m^2 & 0 \\ \alpha & 0 & \boldsymbol{e}_b^2 \end{array}\right), \tag{9.13}$$

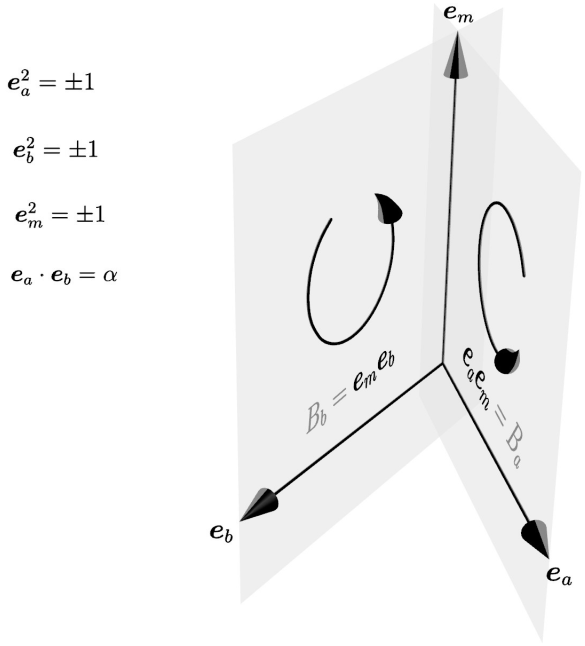
$\boldsymbol{e}_a^2 = \pm 1$
$\boldsymbol{e}_b^2 = \pm 1$
$\boldsymbol{e}_m^2 = \pm 1$
$\boldsymbol{e}_a \cdot \boldsymbol{e}_b = \alpha$
$\boldsymbol{e}_m$
$B_b = \boldsymbol{e}_m\boldsymbol{e}_b$
$\boldsymbol{e}_a\boldsymbol{e}_m = B_a$
$\boldsymbol{e}_b$
$\boldsymbol{e}_a$

Figure 9.1: Basis for Intersecting Planes

where $\alpha = \boldsymbol{e}_a \cdot \boldsymbol{e}_b$[7]. With this notation we have

$$R_a = \mathrm{e}^{\theta_a \boldsymbol{e}_a \boldsymbol{e}_m} = C_a + S_a \boldsymbol{e}_a \boldsymbol{e}_m \tag{9.14}$$
$$R_b = \mathrm{e}^{\theta_b \boldsymbol{e}_m \boldsymbol{e}_b} = C_b + S_b \boldsymbol{e}_m \boldsymbol{e}_b, \tag{9.15}$$

where C_a, S_a, C_b, and S_b are the functions of θ_a or θ_b appropriate for the sign of the square of the bivectors $\boldsymbol{e}_a\boldsymbol{e}_m$ and $\boldsymbol{e}_m\boldsymbol{e}_b$. Now evaluate R_aR_b:

$$R = R_a R_b = S + B \tag{9.16}$$
$$S = \alpha \boldsymbol{e}_m^2 S_a S_b + C_a C_b \tag{9.17}$$
$$B = S_a C_b \boldsymbol{e}_a \boldsymbol{e}_m + \boldsymbol{e}_m^2 S_a S_b \boldsymbol{e}_a \boldsymbol{e}_b + C_a S_b \boldsymbol{e}_m \boldsymbol{e}_b \tag{9.18}$$
$$B^2 = \alpha^2 S_a^2 S_b^2 + 2\alpha \boldsymbol{e}_m^2 S_a S_b C_a C_b - \boldsymbol{e}_b^2 \boldsymbol{e}_m^2 C_a^2 S_b^2 - \boldsymbol{e}_a^2 \boldsymbol{e}_m^2 C_b^2 S_a^2 - \boldsymbol{e}_a^2 \boldsymbol{e}_b^2 S_a^2 S_b^2 \tag{9.19}$$
$$S^2 - B^2 = C_a^2 C_b^2 + \boldsymbol{e}_b^2 \boldsymbol{e}_m^2 C_a^2 S_b^2 + \boldsymbol{e}_a^2 \boldsymbol{e}_m^2 S_a^2 C_b^2 + \boldsymbol{e}_a^2 \boldsymbol{e}_b^2 S_a^2 S_b^2. \tag{9.20}$$

Evaluating $S^2 - B^2$ for all combinations of $\boldsymbol{e}_a^2$, $\boldsymbol{e}_b^2$, and $\boldsymbol{e}_m^2$ and using trigometric identities to simplify $S^2 - B^2$ gives:

---

[7] Some of the nomenclature associated with orthogonality is confusing. If $\alpha = 0$ we say the planes defined by B_a and B_b are orthogonal. However, if B_a and B_b define subspaces with only the zero vector in common we say the subspaces are orthogonal. This is equivalent to $B_a \cdot B_b = 0$.

$\boldsymbol{e}_a^2$	$\boldsymbol{e}_b^2$	$\boldsymbol{e}_m^2$	$S^2 - B^2$
-1	-1	-1	$\sin^2(\theta_a)\sin^2(\theta_b) + \sin^2(\theta_a)\cos^2(\theta_b) + \sin^2(\theta_b)\cos^2(\theta_a) + \cos^2(\theta_a)\cos^2(\theta_b) = 1$
-1	-1	1	$\sinh^2(\theta_a)\sinh^2(\theta_b) - \sinh^2(\theta_a)\cosh^2(\theta_b) - \sinh^2(\theta_b)\cosh^2(\theta_a) + \cosh^2(\theta_a)\cosh^2(\theta_b) = 1$
-1	1	-1	$-\sin^2(\theta_a)\sinh^2(\theta_b) + \sin^2(\theta_a)\cosh^2(\theta_b) - \cos^2(\theta_a)\sinh^2(\theta_b) + \cos^2(\theta_a)\cosh^2(\theta_b) = 1$
-1	1	1	$-\sin^2(\theta_b)\sinh^2(\theta_a) + \sin^2(\theta_b)\cosh^2(\theta_a) - \cos^2(\theta_b)\sinh^2(\theta_a) + \cos^2(\theta_b)\cosh^2(\theta_a) = 1$
1	-1	-1	$-\sin^2(\theta_b)\sinh^2(\theta_a) + \sin^2(\theta_b)\cosh^2(\theta_a) - \cos^2(\theta_b)\sinh^2(\theta_a) + \cos^2(\theta_b)\cosh^2(\theta_a) = 1$
1	-1	1	$-\sin^2(\theta_a)\sinh^2(\theta_b) + \sin^2(\theta_a)\cosh^2(\theta_b) - \cos^2(\theta_a)\sinh^2(\theta_b) + \cos^2(\theta_a)\cosh^2(\theta_b) = 1$
1	1	-1	$\sinh^2(\theta_a)\sinh^2(\theta_b) - \sinh^2(\theta_a)\cosh^2(\theta_b) - \sinh^2(\theta_b)\cosh^2(\theta_a) + \cosh^2(\theta_a)\cosh^2(\theta_b) = 1$
1	1	1	$\sin^2(\theta_a)\sin^2(\theta_b) + \sin^2(\theta_a)\cos^2(\theta_b) + \sin^2(\theta_b)\cos^2(\theta_a) + \cos^2(\theta_a)\cos^2(\theta_b) = 1$

This proves that if the blades B_a and B_b do not commute we can construct a 2-blade B_{ab} such that

$$\mathrm{e}^{B_{a,b}} = \mathrm{e}^{B_a}\,\mathrm{e}^{B_b} = S + B,\quad B_{a,b} = \left\{ \begin{array}{ll} B^2 < 0, & B_{a,b} = \mathrm{Sin}^{-1}\left(\sqrt{-B^2}\right)\dfrac{B}{\sqrt{-B^2}} \\ B^2 > 0, & B_{a,b} = \mathrm{Sinh}^{-1}\left(\sqrt{B^2}\right)\dfrac{B}{\sqrt{B^2}} \end{array} \right\}. \tag{9.21}$$

Now consider the general rotor R that is composed of products of the form

$$R = \mathrm{e}^{B_1}\dots\mathrm{e}^{B_r}. \tag{9.22}$$

where the B_i's are all 2-blades.

Equation (9.22) is reduced to $R = \mathrm{e}^{B}$ where B is a bivector (not necessarily a blade) via the following operations on adjacent terms in the product.

1. If B_iB_{i+1} is grade 4, B_i and B_{i+1} define orthogonal subspaces and e^{B_i} and $\mathrm{e}^{B_{i+1}}$ commute. If needed reverse the order of e^{B_i} and $\mathrm{e}^{B_{i+1}}$ to get one or the other term closer to a term it does not commute with.

2. If B_iB_{i+1} is a scalar (grade 0), B_i and B_{i+1} define the same subspace and commute. Substitute $\mathrm{e}^{B_i+B_{i+1}}$ for $\mathrm{e}^{B_i}\,\mathrm{e}^{B_{i+1}}$ and remove term $\mathrm{e}^{B_{i+1}}$.

3. If B_iB_{i+1} is grade 2, B_i and B_{i+1} define intersecting subspaces and do not commute. Use eq (9.21) to substitue $\mathrm{e}^{B_{i,i+1}}$ for $\mathrm{e}^{B_i}\,\mathrm{e}^{B_{i+1}}$ and remove term $\mathrm{e}^{B_{i+1}}$.

Repeat operations 1, 2, and 3 until R is in the form $R = \mathrm{e}^{B'_1}\dots\mathrm{e}^{B'_r}$ where $s \le r$ and all of the B'_i commute with each other. Then we have

$$R = \mathrm{e}^{B'_1+\dots+B'_s}, \tag{9.23}$$

and the assertion that ever rotation can be expressed as the exponential of a bivector is proved.

9.3.2 Every Exponential of a Bivector is a Rotor

Now we need to prove the converse, that the exponential of a general bivector is the spin representation of a rotation.

Let B be a general bivector and note that $B^\dagger = -B$. Then let $R = \mathrm{e}^B$ so that

$$\left(\mathrm{e}^B\right)^\dagger = \sum_{i=0}^{\infty} \frac{\left(B^i\right)^\dagger}{i!} = \sum_{i=0}^{\infty} \frac{\left(B^\dagger\right)^i}{i!} = \sum_{i=0}^{\infty} \frac{\left(-B\right)^i}{i!} = \mathrm{e}^{-B}\,. \tag{9.24}$$

Likewise since B commutes with itself we have

$$RR^\dagger = \mathrm{e}^B \left(\mathrm{e}^B\right)^\dagger = \mathrm{e}^B\,\mathrm{e}^{-B} = \mathrm{e}^{B-B} = \mathrm{e}^0 = 1. \tag{9.25}$$

Now consider the following function of λ where a_0 is any vector,

$$a\left(\lambda\right) = \mathrm{e}^{-\frac{\lambda B}{2}}\, a_0\, \mathrm{e}^{\frac{\lambda B}{2}} \tag{9.26}$$

and develop the power series expansion for $a\left(\lambda\right)$

$$\begin{aligned} \frac{da}{d\lambda} &= \frac{1}{2}\,\mathrm{e}^{-\frac{\lambda B}{2}} \left(-Ba_0 + a_0 B\right) \mathrm{e}^{\frac{\lambda B}{2}} \\ &= \mathrm{e}^{-\frac{\lambda B}{2}} \left(a_0 \cdot B\right) \mathrm{e}^{\frac{\lambda B}{2}} \qquad (9.27) \\ \frac{d^2 a}{d\lambda^2} &= \mathrm{e}^{-\frac{\lambda B}{2}} \left(\left(a_0 \cdot B\right) \cdot B\right) \mathrm{e}^{\frac{\lambda B}{2}} \qquad (9.28) \\ \frac{d^3 a}{d\lambda^3} &= \mathrm{e}^{-\frac{\lambda B}{2}} \left(\left(\left(a_0 \cdot B\right) \cdot B\right) \cdot B\right) \mathrm{e}^{\frac{\lambda B}{2}}\,. \qquad (9.29) \end{aligned}$$

Thus every derivative $\dfrac{d^r a}{d\lambda^r}$ is a vector since the dot product of a vector and a bivector is always a vector and the Taylor series expansion of eq (9.26) about $\lambda = 0$ is

$$a\left(\lambda\right) = a_0 + a_0 \lambda \cdot B + \frac{\lambda^2}{2!}\left(a_0 \cdot B\right) \cdot B + \cdots\,. \tag{9.30}$$

Thus $a\left(\lambda\right)$ is a vector for any λ (take $\lambda = 1$) and $R = \mathrm{e}^{-\frac{B}{2}}$ and $\underline{R}a_0 = \mathrm{e}^{-\frac{B}{2}}\, a_0\, \mathrm{e}^{\frac{B}{2}}$ so that we have

$$\begin{aligned} \left(\underline{R}a_0\right) \cdot \left(\underline{R}a_0\right) &= \left(\mathrm{e}^{-\frac{B}{2}}\, a_0\, \mathrm{e}^{\frac{B}{2}}\right) \cdot \left(\mathrm{e}^{-\frac{B}{2}}\, a_0\, \mathrm{e}^{\frac{B}{2}}\right) \\ &= \left(\mathrm{e}^{-\frac{B}{2}}\, a_0\, \mathrm{e}^{\frac{B}{2}}\right) \left(\mathrm{e}^{-\frac{B}{2}}\, a_0\, \mathrm{e}^{\frac{B}{2}}\right) \\ &= \left(\mathrm{e}^{-\frac{B}{2}}\, a_0 a_0\, \mathrm{e}^{\frac{B}{2}}\right) \\ &= a_0 a_0 \left(\mathrm{e}^{-\frac{B}{2}}\, \mathrm{e}^{\frac{B}{2}}\right) \\ &= a_0 a_0 = a_0 \cdot a_0. \qquad (9.31) \end{aligned}$$

Thus $e^{-\frac{B}{2}}$ preserves the length of a_0 and thus must be a rotation operator.

9.4 The Grassmann Algebra

Let $\mathcal{V}^n$ be an n-dimensional real vector space with basis $\{\boldsymbol{w}_i\} : 1 \le i \le n$ and $\wedge$ is the outer (wedge) product for the geometric algebra define on $\mathcal{V}^n$. As before the geometric object

$$v_1 \wedge v_2 \wedge \ldots \wedge v_k \tag{9.32}$$

is a k-blade in the Grassmann or the geometric algebra where the v_i's are k independent vectors in $\mathcal{V}_n$ and a linear combination of k-blades is called a k-vector. The space of all k-vectors is just all the k-grade multivectors in the geometric algebra $\mathcal{G}(\mathcal{V}_n)$. Denote the space of all k-vectors in $\mathcal{V}^n$ by $\Lambda_n^k = \Lambda^k(\mathcal{V}^n)$ with

$$\dim\left(\Lambda_n^k\right) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}. \tag{9.33}$$

Letting $\Lambda_n^1 = \mathcal{V}^n$ and $\Lambda_n^0 = \Re$ the entire Grassmann algebra is a 2^n-dimensional space.[8]

$$\Lambda_n = \sum_{k=0}^{n} \Lambda_n^k. \tag{9.34}$$

The Grassmann algebra of a vector space $\mathcal{V}^n$ is denoted by $\Lambda(\mathcal{V}^n)$.

9.5 The Dual Space to $\mathcal{V}_n$

The dual space $(\mathcal{V}^n)^*$ to $\mathcal{V}^n$ is defined as follows. Let $\{\boldsymbol{w}_i\}$ be a basis for $\mathcal{V}^n$ and define the basis, $\{\boldsymbol{w}_i^*\}$ for $(\mathcal{V}^n)^*$ by

$$\boldsymbol{w}_i^* \cdot \boldsymbol{w}_j = \frac{1}{2}\delta_{ij}. \tag{9.35}$$

---

[8] Consider the geometric algebra $\mathcal{G}(\mathcal{V}^n)$ of $\mathcal{V}^n$ with a null basis $\{\boldsymbol{w}_i\} : \boldsymbol{w}_i^2 = 0, 1 \le i \le n$. Then

$$\begin{aligned}
(\boldsymbol{w}_i + \boldsymbol{w}_j)^2 &= \boldsymbol{w}_i^2 + \boldsymbol{w}_i\boldsymbol{w}_j + \boldsymbol{w}_j\boldsymbol{w}_i + \boldsymbol{w}_j^2 \\
0 &= \boldsymbol{w}_i\boldsymbol{w}_j + \boldsymbol{w}_j\boldsymbol{w}_i \\
\boldsymbol{w}_i\boldsymbol{w}_j &= -\boldsymbol{w}_j\boldsymbol{w}_i \\
0 &= 2\boldsymbol{w}_i \cdot \boldsymbol{w}_j.
\end{aligned}$$

If the basis set is null the metric tensor is null and $\boldsymbol{w}_i\boldsymbol{w}_j = \boldsymbol{w}_i \wedge \boldsymbol{w}_j$ even if $i = j$ and the geometric algebra is a Grassmann algebra.

Again the dual space $\mathcal{V}^{n*}$ has its own Grassmann algebra $\Lambda^*\left(\mathcal{V}^n\right)$ given by

$$\Lambda^*\left(\mathcal{V}^n\right) = \Lambda^*_n = \sum_{k=0}^{n} \Lambda^{k*}_n . \tag{9.36}$$

$\Lambda^*\left(\mathcal{V}^n\right)$ can be represented as a geometric algebra by imposing the null metric condition $\left(\boldsymbol{w}^*_i\right)^2 = 0$ as in the case of $\Lambda\left(\mathcal{V}^n\right)$.

9.6 The Mother Algebra

From the base vector space and it's dual space one can construct a $2n$ dimensional vector space from the direct sum of the two vector spaces as defined in Appendix F

$$\Re^{n,n} \equiv \mathcal{V}^n \oplus \left(\mathcal{V}^n\right)^* \tag{9.37}$$

with basis $\left\{\boldsymbol{w}_i, \boldsymbol{w}^*_j\right\}$. An orthogonal basis for $\Re^{n,n}$ can be constructed as follows (using equation 9.35):

$$\begin{aligned}
\boldsymbol{e}_i &= \boldsymbol{w}_i + \boldsymbol{w}^*_i && (9.38)\\
\bar{\boldsymbol{e}}_i &= \boldsymbol{w}_i - \boldsymbol{w}^*_i && (9.39)\\
\boldsymbol{e}_i \cdot \boldsymbol{e}_j &= \left(\boldsymbol{w}_i + \boldsymbol{w}^*_i\right) \cdot \left(\boldsymbol{w}_j + \boldsymbol{w}^*_j\right) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j + \boldsymbol{w}_i \cdot \boldsymbol{w}^*_j + \boldsymbol{w}^*_i \cdot \boldsymbol{w}_j + \boldsymbol{w}^*_i \cdot \boldsymbol{w}^*_j \\
&= \delta_{ij} && (9.40)\\
\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_j &= \left(\boldsymbol{w}_i - \boldsymbol{w}^*_i\right) \cdot \left(\boldsymbol{w}_j - \boldsymbol{w}^*_j\right) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j - \boldsymbol{w}_i \cdot \boldsymbol{w}^*_j - \boldsymbol{w}^*_i \cdot \boldsymbol{w}_j + \boldsymbol{w}^*_i \cdot \boldsymbol{w}^*_j \\
&= -\delta_{ij} && (9.41)\\
\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j &= \left(\boldsymbol{w}_i + \boldsymbol{w}^*_i\right) \cdot \left(\boldsymbol{w}_j - \boldsymbol{w}^*_j\right) \\
&= \boldsymbol{w}_i \cdot \boldsymbol{w}_j - \boldsymbol{w}_i \cdot \boldsymbol{w}^*_j + \boldsymbol{w}^*_i \cdot \boldsymbol{w}_j - \boldsymbol{w}^*_i \cdot \boldsymbol{w}^*_j \\
&= 0. && (9.42)
\end{aligned}$$

Thus $\Re^{n,n}$ can also be represented by the direct sum of an n-dimensional Euclidian vector space, E^n, and a n-dimensional anti-Euclidian vector space (square of all basis vectors is -1) $\bar{E}^n$

$$\Re^{n,n} = E^n \oplus \bar{E}^n . \tag{9.43}$$

In this case E^n and $\bar{E}^n$ are orthogonal[9] since $\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j = 0$. The geometric algebra of $\Re^{n,n}$ is defined and denoted by

$$\Re_{n,n} \equiv \mathcal{G}\left(\Re^{n,n}\right) \tag{9.44}$$

has dimension 2^{2n} with k-vector subspaces $\Re^k_{n,n} = \mathcal{G}^k\left(\Re^{n,n}\right)$ and is called the *mother algebra.*[10]

From the basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_i\}$ we can construct $(p+q)$-blades

$$E_{p,q} = E_p \wedge \bar{E}^\dagger_q = E_p\bar{E}^\dagger_q, \tag{9.45}$$

where[11]

$$E_p = \boldsymbol{e}_{i_1}\boldsymbol{e}_{i_2}\dots\boldsymbol{e}_{i_p} = E_{p,0} \quad 1 \le i_1 < i_2 < \dots < i_p \le n \tag{9.46}$$

$$\bar{E}_q = \bar{\boldsymbol{e}}_{j_1}\bar{\boldsymbol{e}}_{j_2}\dots\bar{\boldsymbol{e}}_{j_q} = \bar{E}_{0,q} \quad 1 \le j_1 < j_2 < \dots < j_q \le n. \tag{9.47}$$

Each blade determines a projection of $\underline{E}_{p,q}$ of $\Re^{n,n}$ into a $(p+q)$-dimensional subspace $\Re^{p,q}$ defined by (see section 5.1.2 and equation 1.42)[12]

$$\underline{E}_{p,q}a \equiv \left(a \cdot E_{p,q}\right)E^{-1}_{p,q} = \frac{1}{2}\left(a - (-1)^{p+q}E_{p,q}aE^{-1}_{p,q}\right). \tag{9.48}$$

A vector, a, is in $\Re^{p,q}$ if and only if

$$a \wedge E_{p,q} = 0 = aE_{p,q} + (-1)^{p+q}E_{p,q}a. \tag{9.49}$$

---

[9] Every vector in E^n is orthogonal to every vector in $\bar{E}^n$.

[10] As an example of the equivalence of $E^n \oplus \bar{E}^n$ and $\mathcal{V}^n \oplus \mathcal{V}^{n*}$ consider the metric tensors of each representation of $\Re^{2,2}$. The metric tensor of $E^2 \oplus \bar{E}^2$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

with eigenvalues $[1, 1, -1, -1]$. Thus the signature of the metric is $(2, 2)$. The metric tensor of $\mathcal{V}^2 \oplus \mathcal{V}^{2*}$ is

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}/2$$

with eigenvalues $[1/2, 1/2, -1/2, -1/2]$. Thus the signature of the metric is $(2, 2)$. As expected both representations have the same signature.

[11] Since the $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_i\}$ form an orthogonal set and specifying that no factors are repeated we can use the geometric product in equations 9.46 and 9.47 instead of the outer (wedge) product.

[12] The underbar notation as in $\underline{E}_{p,q}(a)$ allows one to distinguish linear operators from elements in the algebra such as $E_{p,q}$.

For $p+q=n$, the blade $E_{p,q}$ determines a split of $\Re^{n,n}$ into orthogonal subspaces with *complementary signature*[13], as expressed by

$$\Re^{n,n} = \Re^{p,q} \oplus \bar{\Re}^{p,q}. \tag{9.50}$$

For the case of $q=0$ equation 9.48 can be written as

$$\underline{E}_n a = \frac{1}{2}\left(a + a^*\right), \tag{9.51}$$

where a^* is defined by

$$a^* \equiv (-1)^{n+1} E_n a E_n^{-1}. \tag{9.52}$$

It follows immediately that $\boldsymbol{e}_i^* = \boldsymbol{e}_i$ and $\left(\bar{\boldsymbol{e}}_i\right)^* = -\bar{\boldsymbol{e}}_i$.[14] The split of $\Re^{n,n}$ given by equation 9.37 cannot be constructed as the split in equation 9.43 since the vectors expanded in the $\left\{\boldsymbol{w}_i, \boldsymbol{w}_j^*\right\}$ basis cannot be normalized since they are null vectors. Instead consider a bivector[15] K in $\Re^{n,n}$

$$K = \sum_{i=1}^{n} K_i, \tag{9.53}$$

where the K_i are distinct commuting blades, $K_i \times K_j = 0$ (commutator product), normalized to $K_i^2 = 1$. The bivector K defines the automorphism $\underline{K} : \Re^{n,n} \rightarrow \Re^{n,n}$

$$\bar{a} = \underline{K} a \equiv a \times K = a \cdot K. \tag{9.54}$$

This maps every vector a into the vector $\bar{a}$ which is called the *complement* of a with respect to K.

Each K_i is a bivector blade that defines a two dimensional Minkowski (since $K_i^2 = 1$) subspace of $\Re^{n,n}$. Since the blades commute, $K_i \times K_j = 0$, they define disjoint subspaces of $\Re^{n,n}$ and since

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[13] The signature of $\Re^{p,q}$ is (p,q) as opposed to $\bar{\Re}^{p,q}$ with signature (q,p).

[14] This is obvious by

$$\begin{aligned}
\boldsymbol{e}_i^* &= (-1)^{n+1} E_n \boldsymbol{e}_i E_n^{-1} = (-1)^{n+1} (-1)^{n-1} \boldsymbol{e}_i E_n E_n^{-1} = (-1)^{2n} \boldsymbol{e}_i = \boldsymbol{e}_i, \\
\left(\bar{\boldsymbol{e}}_i\right)^* &= (-1)^{n+1} E_n \left(\bar{\boldsymbol{e}}_i\right) E_n^{-1} = (-1)^{n+1} (-1)^{n} \left(\bar{\boldsymbol{e}}_i\right) E_n E_n^{-1} = (-1)^{2n+1} \left(\bar{\boldsymbol{e}}_i\right) = -\left(\bar{\boldsymbol{e}}_i\right).
\end{aligned}$$

[15] In general the basis blades for bivectors in $\Re_{n,n}$ do not commute since the dimension of the bivector subspace, $\Re_{n,n}^2$ is

$$\binom{2n}{2} = \frac{(2n)!}{2!\,(2n-2)!} = n\,(2n-1) = 2n^2 - n.$$

If one has more than n bivector blades the planes defined by at least two of the blades will intersect.

there are n of them they span $\Re^{n,n}$. Since $K_i^2 = 1$ there exists an orthonormal Minkowski basis $\boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}_i$ such that

$$\begin{aligned} \boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j &= 0, && (9.55)\\ \boldsymbol{e}_i \cdot \boldsymbol{e}_i &= -\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_i = 1. && (9.56) \end{aligned}$$

Then if $a \in \Re^{n,n}$ and using equations 9.55, 9.56, and from appendix B equations B.2 and B.5 we have

$$\begin{aligned} a \cdot K_i &= a \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i) = -(a \cdot \bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i + (a \cdot \boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i, && (9.57)\\ (a \cdot K_i) \cdot K_i &= (a \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i)) \cdot (\boldsymbol{e}_i \bar{\boldsymbol{e}}_i)\\ &= (a \cdot \boldsymbol{e}_i)\,\boldsymbol{e}_i - (a \cdot \bar{\boldsymbol{e}}_i)\,\bar{\boldsymbol{e}}_i\\ &= \left(a \cdot \boldsymbol{e}^i\right)\boldsymbol{e}_i + \left(a \cdot \bar{\boldsymbol{e}}^i\right)\bar{\boldsymbol{e}}_i = a, && (9.58)\\ (a \cdot K_i) \cdot K_j &= ((a \cdot \bar{\boldsymbol{e}}_i)(\boldsymbol{e}_i \cdot \bar{\boldsymbol{e}}_j) - (a \cdot \boldsymbol{e}_i)(\bar{\boldsymbol{e}}_i \cdot \bar{\boldsymbol{e}}_j))\,\boldsymbol{e}_j && (9.59)\\ &\quad + ((a \cdot \boldsymbol{e}_i)(\bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j) - (a \cdot \bar{\boldsymbol{e}}_i)(\boldsymbol{e}_i \cdot \boldsymbol{e}_j))\,\bar{\boldsymbol{e}}_j = 0,\ \ \forall\, i \neq j \end{aligned}$$

Then

$$\underline{K}a = \sum_{i=1}^{n} a \cdot K_i = \sum_{i=1}^{n} ((a \cdot \boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (a \cdot \bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i) \tag{9.60}$$

and[16]

$$\begin{aligned} \underline{K}^2 a &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a \cdot K_i \right) \cdot K_j\\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} (a \cdot K_i) \cdot K_j\\ &= \sum_{i=1}^{n} (a \cdot K_i) \cdot K_i\\ &= \sum_{i=1}^{n} ((a \cdot \boldsymbol{e}_i)\,\boldsymbol{e}_i - (a \cdot \bar{\boldsymbol{e}}_i)\,\bar{\boldsymbol{e}}_i)\\ &= \sum_{i=1}^{n} \left(\left(a \cdot \boldsymbol{e}^i\right)\boldsymbol{e}_i + \left(a \cdot \bar{\boldsymbol{e}}^i\right)\bar{\boldsymbol{e}}_i \right)\\ &= a. && (9.61) \end{aligned}$$

---

[16] Remember that since K_i defines a Minkowski subspace we have for the reciprocal basis $\boldsymbol{e}^i = \boldsymbol{e}_i$ and $\bar{\boldsymbol{e}}^i = -\bar{\boldsymbol{e}}_i$.

Also

$$\begin{aligned}
a\cdot\bar{a} &= \sum_{i=1}^{n} a\cdot(a\cdot K_i) \\
&= \sum_{i=1}^{n} a\cdot\left((a\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i-(a\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right) \\
&= \sum_{i=1}^{n}\left((a\cdot\boldsymbol{e}_i)\,(a\cdot\bar{\boldsymbol{e}}_i)-(a\cdot\bar{\boldsymbol{e}}_i)\,(a\cdot\boldsymbol{e}_i)\right) \\
&= 0, &(9.62)\\
a^2 &= \sum_{i=1}^{n}\left((a\cdot\boldsymbol{e}_i)^2-(a\cdot\bar{\boldsymbol{e}}_i)^2\right), &(9.63)\\
\bar{a}^2 &= \sum_{i=1}^{n}\left((a\cdot\bar{\boldsymbol{e}}_i)^2-(a\cdot\boldsymbol{e}_i)^2\right), &(9.64)\\
a^2+\bar{a}^2 &= 0. &(9.65)
\end{aligned}$$

Thus defining

$$a_\pm \equiv a\pm\bar{a} = a\pm\underline{K}a = a\pm a\cdot K, \tag{9.66}$$

we have

$$\begin{aligned}
(a_\pm)^2 &= 0, &(9.67)\\
a_+\cdot a_- &= a^2-\bar{a}^2 = 2\sum_{i=1}^{n}(a\cdot\boldsymbol{e}_i)^2. &(9.68)
\end{aligned}$$

Thus the sets $\{a_+\}$ and $\{a_-\}$ of all such vectors are in dual n-dimensional vector spaces ($a_+\in\mathcal{V}^n$ and $a_-\in\mathcal{V}^{n*}$), so K determines the desired null space decomposition of the form in equation 9.37 without referring to a vector basis.[17]

The K for a given $\Re^{n,n}$ is constructed from the basis given in equations 9.38 and 9.39. Then we have from equation 9.60

$$\begin{aligned}
\underline{K}\boldsymbol{e}_i &= \bar{\boldsymbol{e}}_i, &(9.69)\\
\underline{K}\bar{\boldsymbol{e}}_i &= \boldsymbol{e}_i. &(9.70)
\end{aligned}$$

---

[17] The properties $K_i\times K_j=0$ and $K_i^2=1$ allows us to construct K from the basis $\{\boldsymbol{e}_i,\bar{\boldsymbol{e}}_i\}$, but do not specify any particular basis.

Also from equation 9.60

$$\begin{aligned}
\underline{K}\boldsymbol{w}_i &= (\boldsymbol{w}_i\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (\boldsymbol{w}_i\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i, \\
&= \frac{1}{2}\left(((\boldsymbol{e}_i+\bar{\boldsymbol{e}}_i)\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - ((\boldsymbol{e}_i+\bar{\boldsymbol{e}}_i)\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right), \\
&= \frac{1}{2}\left(\bar{\boldsymbol{e}}_i+\boldsymbol{e}_i\right), \\
&= \boldsymbol{w}_i, \qquad (9.71) \\
\underline{K}\boldsymbol{w}_i^* &= (\boldsymbol{w}_i^*\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - (\boldsymbol{w}_i^*\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i, \\
&= \frac{1}{2}\left(((\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i)\cdot\boldsymbol{e}_i)\,\bar{\boldsymbol{e}}_i - ((\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i)\cdot\bar{\boldsymbol{e}}_i)\,\boldsymbol{e}_i\right), \\
&= -\frac{1}{2}\left(\boldsymbol{e}_i-\bar{\boldsymbol{e}}_i\right), \\
&= -\boldsymbol{w}_i^*. \qquad (9.72)
\end{aligned}$$

The basis $\{\boldsymbol{w}_i, \boldsymbol{w}_i^*\}$ is called a *Witt basis* in the theory of quadratic forms.

9.7 The General Linear Group as a Spin Group

We will now use the results in section 1.15 and the extension of a linear vector function to blades (equation 1.77)[18] and the geometric algebra definition of the determinant of a linear transformation (equation 1.79) [19].

We are concerned here with linear transformations on $\Re^{n,n}$ and its subspaces, especially orthogonal transformations. An orthogonal transformation $\underline{R}$ is defined by the property

$$(\underline{R}a)\cdot(\underline{R}b) = a\cdot b \qquad (9.73)$$

$\underline{R}$ is called a rotation if $\det(\underline{R}) = 1$, that is, if

$$\underline{R}E_{n,n} = E_{n,n}, \qquad (9.74)$$

where $E_{n,n} = E_n\bar{E}_n^\dagger$ is the pseudoscalar for $\Re_{n,n}$ (equation 9.45). These rotations form a group called the *special orthogonal group* SO(n).

From section 1.10 we know that every rotation can be expressed by the canonical form

$$\underline{R}a = RaR^\dagger, \qquad (9.75)$$

---

[18] $\underline{f}(a\wedge b\wedge\ldots) = \underline{f}(a)\wedge\underline{f}(a)\wedge\ldots$

[19] $\underline{f}(I) = \det\left(\underline{f}\right)I$ where I is the pseudoscalar.

where R is an even multivector (*rotor*) satisfying

$$RR^{\dagger} = 1. \tag{9.76}$$

The rotors form a multiplicative group called the *spin group* or *spin representation* of SO(n), and it is denoted by Spin(n). Spin(n) is said to be a double covering of SO(n), since equation 9.75 shows that both $\pm R$ correspond to the same $\underline{R}$.

From equation 9.76 it follows that $R^{-1} = R^{\dagger}$ and that the inverse of the rotation is

$$\underline{R}^{\dagger} a = R^{\dagger} a R. \tag{9.77}$$

This implies that from the definition of the adjoint (using **RR5** in appendix C)

$$a \cdot (\underline{R} b) = \left\langle a R b R^{\dagger} \right\rangle = \left\langle b R^{\dagger} a R \right\rangle = b \cdot \left(\underline{R}^{\dagger} a\right). \tag{9.78}$$

The adjoint of a rotation is equal to its inverse.

> "It can be shown that every rotor can be expressed in exponential form
>
> $$R = \pm\, \mathrm{e}^{\frac{1}{2}B}, \text{ with } R^{\dagger} = \pm\, \mathrm{e}^{-\frac{1}{2}B}, \tag{9.79}$$
>
> where B is a bivector (section 1.10.2) called the generator of R or $\underline{R}$, and the minus sign can usually be eliminated by a change in the definition of B. Thus every bivector determines a unique rotation. The bivector generators of a spin or rotation group form a Lie algebra under the commutator product. **This reduces the description of Lie groups to Lie algebras.** The Lie algebra of SO(n) and Spin(n) is designated by $\mathfrak{so}(n)$. It consists of the entire bivector space $\Re^2_{n,n}$. Our task will be to prove that and develop a systematic way to find them.
>
> Lie groups are classified according to their invariants. For the *classical groups* the invariants are nondegenerate bilinear (quadratic) forms. Geometric algebra supplies us with a simpler alternative of invariants, namely, the multivectors which determine the bilinear forms. As emphasized in reference [4], every bilinear form can be written as $a \cdot \left(\underline{Q}(b)\right)$ where $\underline{Q}$ is a linear operator, and the form is nondegenerate if $\underline{Q}$ is nonsingular (i.e., det $\left(\underline{Q}\right) \neq 0$)." [20]

To prove that $\mathrm{e}^{\frac{B}{2}}$ is a rotation if B is a general bivector consider the function[21]

$$a(\lambda) = \mathrm{e}^{\frac{\lambda B}{2}}\, a\, \mathrm{e}^{-\frac{\lambda B}{2}} \tag{9.80}$$

---

[20] From reference [5].

[21] We let $a = a(0)$.

where a is a vector. Then differentiate $a(\lambda)$

$$\begin{aligned}\frac{da}{d\lambda} &= \frac{B}{2}\,\mathrm{e}^{\frac{\lambda B}{2}}\,a\,\mathrm{e}^{-\frac{\lambda B}{2}} - \mathrm{e}^{\frac{\lambda B}{2}}\,a\,\mathrm{e}^{-\frac{\lambda B}{2}}\,\frac{B}{2} \\ &= \frac{B}{2}a(\lambda) - a(\lambda)\,\frac{B}{2} \\ &= B\cdot a(\lambda) \qquad (9.81)\\ \frac{d^2a}{d\lambda^2} &= B\cdot\left(\frac{da}{d\lambda}\right) = B\cdot\left(B\cdot a(\lambda)\right) \qquad (9.82)\\ \left.\frac{d^ra}{d\lambda^r}\right|_{\lambda=0} &= \underbrace{B\cdot\left(B\cdot\left(\ldots\left(B\cdot a\right)\right)\ldots\right)}_{r \text{ copies of } B}. \qquad (9.83)\end{aligned}$$

so that the Taylor expansion of $a(1)$ is

$$\mathrm{e}^{\frac{B}{2}}\,a\,\mathrm{e}^{-\frac{B}{2}} = a + B\cdot a + \frac{1}{2!}B\cdot(B\cdot a) + \frac{1}{3!}B\cdot(B\cdot(B\cdot a)) + \cdots . \qquad (9.84)$$

Since every term in eq (9.84) is a vector then $\mathrm{e}^{\frac{B}{2}}\,a\,\mathrm{e}^{-\frac{B}{2}}$ is a vector. Finally let a and b be vectors then (using RR5 in appendix C)

$$\begin{aligned}\left(\mathrm{e}^{\frac{B}{2}}\,a\,\mathrm{e}^{-\frac{B}{2}}\right)\cdot\left(\mathrm{e}^{\frac{B}{2}}\,b\,\mathrm{e}^{-\frac{B}{2}}\right) &= \left\langle \mathrm{e}^{\frac{B}{2}}\,a\,\mathrm{e}^{-\frac{B}{2}}\,\mathrm{e}^{\frac{B}{2}}\,b\,\mathrm{e}^{-\frac{B}{2}}\right\rangle \\ &= \left\langle \mathrm{e}^{\frac{B}{2}}\,ab\,\mathrm{e}^{-\frac{B}{2}}\right\rangle \\ &= \left\langle \mathrm{e}^{-\frac{B}{2}}\,\mathrm{e}^{\frac{B}{2}}\,ab\right\rangle \\ &= \langle ab\rangle = a\cdot b. \qquad (9.85)\end{aligned}$$

Thus $\mathrm{e}^{\frac{B}{2}}$ is a rotor since $\mathrm{e}^{\frac{B}{2}}\,a\,\mathrm{e}^{-\frac{B}{2}}$ is a vector and the transformation generated by $\mathrm{e}^{\frac{B}{2}}$ preserves the inner product of two vectors.

For an n-dimensional vector space the number of linearly independent bivectors is $\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n^2-n}{2}$, which is also the number of generators for the spin group $Spin(n)$ which are $\mathrm{e}^{\frac{\theta_{ij}\boldsymbol{e}_i\wedge\boldsymbol{e}_j}{2}} = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)\boldsymbol{e}_i\wedge\boldsymbol{e}_j \ \ \forall\ \ 1\le i<j\le n$ where θ_{ij} is the rotation in the plane defined by $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$. Thus there is a one to one correspondence between the bivectors, B, and the rotations in an n-dimensional vector space.

$\underline{Q}$ is invariant under a rotation $\underline{R}$ if

$$\left(\underline{R}a\right)\cdot\left(\underline{Q}\,\underline{R}b\right) = a\cdot\left(\underline{Q}b\right). \qquad (9.86)$$

Using equation 9.78 transforms equation 9.86 to

$$\begin{aligned}(\underline{R}a)\cdot\left(\underline{Q}\underline{R}b\right) &= a\cdot\left(\underline{Q}b\right),\\ \left(\underline{Q}\underline{R}b\right)\cdot(\underline{R}a) &= a\cdot\left(\underline{Q}b\right),\\ a\cdot\left(\underline{R}^{\dagger}\left(\underline{Q}\underline{R}b\right)\right) &= a\cdot\left(\underline{Q}b\right),\\ a\cdot\left(\underline{R}^{\dagger}\underline{Q}\underline{R}b\right) &= a\cdot\left(\underline{Q}b\right), \qquad (9.87)\end{aligned}$$

or since the a and b in equation 9.87 are arbitrary

$$\begin{aligned}\underline{R}^{\dagger}\underline{Q}\underline{R} &= \underline{Q} = \underline{R}\underline{Q}\underline{R}^{\dagger}, \qquad (9.88)\\ \underline{R}\underline{R}^{\dagger}\underline{Q}\underline{R} &= \underline{R}\underline{Q}, \qquad (9.89)\\ \underline{Q}\underline{R} &= \underline{R}\underline{Q}. \qquad (9.90)\end{aligned}$$

Thus the invariance group of consists of those rotations which commute with $\underline{Q}$.

This is obviously a group since if

$$\begin{aligned}\underline{R}_1\underline{Q} &= \underline{Q}\underline{R}_1, \qquad (9.91)\\ \underline{R}_2\underline{Q} &= \underline{Q}\underline{R}_2, \qquad (9.92)\end{aligned}$$

then

$$\begin{aligned}\underline{R}_1\underline{R}_2\underline{Q} &= \underline{R}_1\underline{Q}\underline{R}_2, \qquad (9.93)\\ \underline{R}_1\underline{R}_2\underline{Q} &= \underline{Q}\underline{R}_1\underline{R}_2. \qquad (9.94)\end{aligned}$$

As a specific case consider the quadratic form where $\underline{Q}b = b^{*}$. Then

$$\begin{aligned}\underline{Q}b &= (-1)^{n+1}E_nbE_n^{-1}, \qquad (9.95)\\ \underline{Q}\underline{R}b &= (-1)^{n+1}E_nRbR^{\dagger}E_n^{-1},\\ \underline{R}\underline{Q}b &= (-1)^{n+1}RE_nbE_n^{-1}R^{\dagger},\\ E_nRbR^{\dagger}E_n^{-1} &= RE_nbE_n^{-1}R^{\dagger},\\ E_nR &= RE_n,\\ E_n &= RE_nR^{\dagger}. \qquad (9.96)\end{aligned}$$

Thus, the invariance of the bilinear form $a\cdot b^{*}$ is equivalent to the invariance of the n-blade E_n.

We can determine a representation for R by noting the generators of any rotation in $\Re^{n,n}$ can be written in the form

$$\begin{aligned} R &= \mathrm{e}^{\frac{\theta}{2}\boldsymbol{u}\boldsymbol{v}}, \\ &= \left\{ \begin{array}{c} \cos(\theta/2) \\ \cosh(\theta/2) \end{array} \right\} + \left\{ \begin{array}{c} \sin(\theta/2) \\ \sinh(\theta/2) \end{array} \right\} \boldsymbol{u}\boldsymbol{v}, \\ &= \left\{ \begin{array}{c} \cos(\theta/2) \\ \cosh(\theta/2) \end{array} \right\} + \left\{ \begin{array}{c} \sin(\theta/2) \\ \sinh(\theta/2) \end{array} \right\} \sum_{i<j} \left(u^i \boldsymbol{e}_i + \bar{u}^i \bar{\boldsymbol{e}}_i \right) \left(v^j \boldsymbol{e}_j + \bar{v}^j \bar{\boldsymbol{e}}_j \right), \end{aligned} \tag{9.97}$$

where $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and $\left|(\boldsymbol{u}\boldsymbol{v})^2\right| = 1$. Equation (9.97) is what makes corresponding the Lie algebras with the bivector commuatator algebra possible and also allows one to calculate the generators of the corresponding Lie group. The generators of the most general rotation in $\Re^{n,n}$ are $\boldsymbol{e}_i\boldsymbol{e}_j$, $\bar{\boldsymbol{e}}_i\boldsymbol{e}_j$, and $\bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j$. The question is which of these generators commute with E_n. The answer is

$$\boldsymbol{e}_i\boldsymbol{e}_j E_n = E_n \boldsymbol{e}_i\boldsymbol{e}_j \tag{9.98}$$
$$\boldsymbol{e}_i\bar{\boldsymbol{e}}_j E_n = -E_n \boldsymbol{e}_i\bar{\boldsymbol{e}}_j \tag{9.99}$$
$$\bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j E_n = E_n \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j. \tag{9.100}$$

Equation 9.100 is obvious since $\bar{\boldsymbol{e}}_i$ and $\bar{\boldsymbol{e}}_j$ give the same number of sign flips, n, in passing through E_n totalling $2n$ an even number. Likewise, in equation 9.98 as $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ give the same number of sign flips, $n-1$, in passing through E_n totalling $2(n-1)$ an even number. In equation 9.99 $\boldsymbol{e}_i$ produces $n-1$ sign flips in traversing E_n and $\bar{\boldsymbol{e}}_j$ produces n sign flips for the same traverse so that the total number of sign flips is $2n-1$ and odd number.

Thus the reqired rotation generators for R are

$$\boldsymbol{e}_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j, \text{ for } i < j = 1, \ldots, n, \tag{9.101}$$
$$\bar{\boldsymbol{e}}_{ij} = \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j, \text{ for } i < j = 1, \ldots, n. \tag{9.102}$$

Also note that since $\boldsymbol{e}_{ij} \times \bar{\boldsymbol{e}}_{kl} = 0$, the barred and unbarred generators commute. Any generator in the algebra can be written in the form

$$B = \boldsymbol{\alpha} : \boldsymbol{e} + \boldsymbol{\beta} : \bar{\boldsymbol{e}}, \tag{9.103}$$

where

$$\boldsymbol{\alpha} : \boldsymbol{e} \equiv \sum_{i<j} \alpha^{ij} \boldsymbol{e}_{ij}, \tag{9.104}$$

and the α^{ij} are scalar coefficients. So that $\boldsymbol{\alpha} : \boldsymbol{e}$ is the generator for any rotation in $\Re^n$ and $\boldsymbol{\beta} : \bar{\boldsymbol{e}}$ is the generator for any rotation in $\bar{\Re}^n$. The corresponding group rotor is (we can factor the exponent since $\boldsymbol{e}$ and $\bar{\boldsymbol{e}}$ generators commute)

$$R = \mathrm{e}^{\frac{1}{2}(\boldsymbol{\alpha}:\boldsymbol{e}+\boldsymbol{\beta}:\bar{\boldsymbol{e}})} = \mathrm{e}^{\frac{1}{2}\boldsymbol{\alpha}:\boldsymbol{e}}\,\mathrm{e}^{\frac{1}{2}\boldsymbol{\beta}:\bar{\boldsymbol{e}}}\,. \tag{9.105}$$

This is the spin representation of the product group $\mathrm{SO}(n)\otimes\mathrm{SO}(n)$.

In most cases the generators of the invariance group are not as obvious and in the case of the $a\cdot b^*$ form. Thus we need some general methods for such determinations. Consider a skew-symmetric bilinear form $\underline{Q}$[22]

$$a\cdot\left(\underline{Q}b\right) = -b\cdot\left(\underline{Q}a\right). \tag{9.106}$$

The form $\underline{Q}$ can be written

$$a\cdot\left(\underline{Q}b\right) = a\cdot(b\cdot Q) = (a\wedge b)\cdot Q, \tag{9.107}$$

where Q is a bivector.[23] We say that the bivector Q in *involutory* if $\underline{Q}$ is nonsingular and

$$\underline{Q}^2 = \pm\underline{1}. \tag{9.108}$$

Note that the operator equation 9.108 only applies to vectors.

Note that

$$\begin{aligned}(\underline{R}a\wedge\underline{R}b)\cdot Q &= \underline{R}\,(a\wedge b)\cdot Q \\ &= \left(R\,(a\wedge b)\,R^\dagger\right)\cdot Q \\ &= \left\langle R\,(a\wedge b)\,R^\dagger Q\right\rangle \\ &= \left\langle (a\wedge b)\,R^\dagger QR\right\rangle \\ &= (a\wedge b)\cdot\left(\underline{R}^\dagger Q\right).\end{aligned} \tag{9.109}$$

---

[22] This selection is done with malice aforethought. To generate the memembers of $GL\,(n,\Re)$ we do not need the most general linear transformation on $\Re^{n,n}$ since the dimension of that group is $4n^2$ and not the n^2 of $GL\,(n,\Re)$.

[23] This is obvious from the properties of the bilinear form that if $a\cdot\left(\underline{Q}b\right) = (a\wedge b)\cdot Q$ then $a\cdot\left(\underline{Q}b\right)$ is linear in a and b and is skew-symmetric

$$\begin{aligned}b\cdot\left(\underline{Q}a\right) &= (b\wedge a)\cdot Q \\ &= -\,(a\wedge b)\cdot Q \\ &= -a\cdot\left(\underline{Q}b\right)\end{aligned}$$

Finally the maximum number of free parameters (coefficients) for the bivector, Q, in an n-dimensional space is $\binom{n}{2} = \frac{n(n-1)}{2}$. This is also the number of independent coefficients in the $n\times n$ antisymmetric matrix that represents the skew-symmetric bilinear form.

Then equation 9.107 gives for a stability condition

$$\begin{aligned}
(\underline{R}a)\cdot\left(\underline{Q}\,\underline{R}b\right) &= a\cdot\left(\underline{Q}b\right),\\
\left((\underline{R}a)\cdot(\underline{R}b)\right)\cdot Q &= (a\wedge b)\cdot Q,\\
\left((\underline{R}a)\wedge(\underline{R}b)\right)\cdot Q &=\\
\underline{R}\,(a\wedge b)\cdot Q &=\\
\left\langle R\,(a\wedge b)\,R^{\dagger}Q\right\rangle &=\\
\left\langle (a\wedge b)\,R^{\dagger}QR\right\rangle &=\\
(a\wedge b)\cdot\left(R^{\dagger}QR\right) &= (a\wedge b)\cdot Q,\\
R^{\dagger}QR &= Q.\\
QR &= RQ \qquad (9.110)
\end{aligned}$$

From equation 9.110 we have that generators of the stability group $G(Q)$ for Q must commute with Q. To learn more about this requirement, we study the commutator of Q with an arbitrary bivector blade $a\wedge b$. Since $a\wedge b = a\times b$ and $a\times Q = a\cdot Q$ the Jacobi identity gives

$$\begin{aligned}
(a\wedge b)\times Q &= (a\times b)\times Q,\\
&= (a\times Q)\times b + a\times(b\times Q),\\
&= (a\times Q)\wedge b + a\wedge(b\times Q),\\
&= (a\cdot Q)\wedge b + a\wedge(b\cdot Q),\\
&= \left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right), \qquad (9.111)
\end{aligned}$$

and then

$$\left((a\wedge b)\times Q\right)\times Q = \left(\left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right)\right)\times Q. \qquad (9.112)$$

Now using the Jacobi identity and the extension of linear functions to blades we have

$$\begin{aligned}
\left(\left(\underline{Q}a\right)\wedge b\right)\times Q &= \left(\left(\underline{Q}a\right)\times b\right)\times Q,\\
&= \left(\underline{Q}a\right)\times(b\times Q) - b\times\left(\left(\underline{Q}a\right)\times Q\right),\\
&= \left(\underline{Q}a\right)\wedge(b\cdot Q) - b\wedge\left(\left(\underline{Q}a\right)\cdot Q\right),\\
&= \left(\underline{Q}a\right)\wedge\left(\underline{Q}b\right) - b\wedge\left(\underline{Q}^2a\right),\\
&= \left(\underline{Q}^2a\right)\wedge b + \underline{Q}\,(a\wedge b). \qquad (9.113)
\end{aligned}$$

Similarly

$$\left(a\wedge\left(\underline{Q}b\right)\right)\times Q = \underline{Q}\,(a\wedge b) + a\wedge\left(\underline{Q}^2b\right), \qquad (9.114)$$

so that, since Q is involutary $\left(\underline{Q}^2 = \pm\underline{1}\right)$

$$\begin{aligned}((a\wedge b)\times Q)\times Q &= \left(\underline{Q}^2 a\right)\wedge b + 2\underline{Q}\,(a\wedge b) + a\wedge\left(\underline{Q}^2 b\right),\\ &= 2\left(\underline{Q}\,(a\wedge b)\pm a\wedge b\right). \qquad (9.115)\end{aligned}$$

By linearity and superposition since equation 9.115 holds for any blade $a\wedge b$ it also holds for any bivector (superposition of 2-blades) B so that

$$(B\times Q)\times Q = 2\left(\underline{Q}B \pm B\right). \qquad (9.116)$$

If B commutes with Q then $B\times Q = 0$ and

$$\begin{aligned}\underline{Q}B \pm B &= 0, &(9.117)\\ \underline{Q}B &= \mp B. &(9.118)\end{aligned}$$

Thus the generators of $G\,(Q)$ are the eigenbivectors of $\underline{Q}$ with eigenvalues ∓ 1.

Now define the bivectors $E^{\pm}\,(a,b)$ and $F\,(a,b)$ by

$$\begin{aligned}E^{\pm}\,(a,b) &\equiv a\wedge b \pm \left(\underline{Q}a\right)\left(\underline{Q}b\right), &(9.119)\\ F\,(a,b) &\equiv \left(\underline{Q}a\right)\wedge b - a\wedge\left(\underline{Q}b\right). &(9.120)\end{aligned}$$

Then using equation 9.111 we get

$$\begin{aligned}E^{\pm}\,(a,b)\times Q &= \left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right) \mp \left(\underline{Q}^2 a\right)\wedge\left(\underline{Q}b\right) \mp \left(\underline{Q}a\right)\wedge\left(\underline{Q}^2 b\right),\\ &= \left(\underline{Q}a\right)\wedge b + a\wedge\left(\underline{Q}b\right) - a\wedge\left(\underline{Q}b\right) - \left(\underline{Q}a\right)\wedge b,\\ &= 0, &(9.121)\\ F\,(a,b)\times Q &= \left(\underline{Q}a\right)\wedge\left(\underline{Q}b\right) + a\wedge\left(\underline{Q}^2 b\right) - \left(\underline{Q}^2 a\right)\wedge b - \left(\underline{Q}a\right)\wedge\left(\underline{Q}b\right),\\ &= \pm a\wedge b \mp a\wedge b,\\ &= 0. &(9.122)\end{aligned}$$

Thus $E^{\pm}\,(a,b)$ and $F\,(a,b)$ are the generators of the stability qroup for Q. A basis for the Lie algebra is obtained by inserting basis vectors for a and b. The commutation relations for the generators $E^{\pm}\,(a,b)$ and $F\,(a,b)$ can be found from equations 9.119, 9.120, and B.13. Evaluation of the commutation relations is simplified by using the eigenvectors of $\underline{Q}$ for a basis, so it is best to defer the task until $\underline{Q}$ is completely specified.

As a concrete example let $Q = K$ where K is from equation 9.53 and use equations 9.69 and 9.70 to get[24]

$$\begin{aligned} E_{ij} = E^{+}\left(\boldsymbol{e}_i, \boldsymbol{e}_j\right) &= \boldsymbol{e}_i \wedge \boldsymbol{e}_j - \left(\underline{K}\boldsymbol{e}_i\right) \wedge \left(\underline{K}\boldsymbol{e}_i\right), \\ &= \boldsymbol{e}_i\boldsymbol{e}_j - \bar{\boldsymbol{e}}_i\bar{\boldsymbol{e}}_j \;\left(i < j\right), \quad (9.123) \\ F_{ij} = F\left(\boldsymbol{e}_i, \boldsymbol{e}_j\right) &= \boldsymbol{e}_i \wedge \left(\underline{K}\boldsymbol{e}_j\right) - \left(\underline{K}\boldsymbol{e}_i\right) \wedge \boldsymbol{e}_j, \\ &= \boldsymbol{e}_i\bar{\boldsymbol{e}}_j - \bar{\boldsymbol{e}}_i\boldsymbol{e}_j \;\left(i < j\right), \quad (9.124) \\ K_i = \frac{1}{2}F_{ii} &= \boldsymbol{e}_i\bar{\boldsymbol{e}}_i. \quad (9.125) \end{aligned}$$

At this point we should note that if a bivector B is a linear combination of E_{ij}, F_{ij}, and F_i, B will commute with Q. Then $R = \mathrm{e}^{\frac{B}{2}}$ also commutes with Q since $\mathrm{e}^{\frac{B}{2}}$ only can contain powers of B. Also E_{ij}, F_{ij}, and F_i are called the generators of the Lie algebra associated with the bilinear form defined by $\underline{K}$ and the number of generators are $\frac{n^2-n}{2} + \frac{n^2-n}{2} + n = n^2$.

The structure equations for the Lie algebra (non-zero commutators of the Lie algebra generators) of the bilinear form $\underline{K}$ are

$$\begin{aligned} E_{ij} \times F_{ij} &= 2\left(K_i - K_j\right) && (9.126) \\ E_{ij} \times K_i &= -F_{ij} && (9.127) \\ F_{ij} \times K_i &= -E_{ij} && (9.128) \\ E_{ij} \times E_{il} &= -E_{jl} && (9.129) \\ F_{ij} \times F_{il} &= E_{jl} && (9.130) \\ F_{ij} \times E_{il} &= F_{jl}. && (9.131) \end{aligned}$$

The structure equations close the algebra with respect to the commutator product (see appendix G for how to calculate the structure equations).

Thus (using the notation of eq (9.104)[25]) the rotors for the stablity group of $\underline{K}$ can be written

$$\mathrm{e}^{\frac{B}{2}} = \mathrm{e}^{\frac{1}{2}\left(\boldsymbol{\alpha}:\boldsymbol{E}+\boldsymbol{\beta}:\boldsymbol{F}+\boldsymbol{\gamma}:\boldsymbol{K}\right)}, \quad (9.132)$$

where the n^2 coefficients are α_{ij}, β_{ij}, and γ_i.

The stability group of K can be identified with the *general linear group* $\mathrm{GL}(n, \Re)$. First we must show that $\underline{K}$ does not mix the subspaces $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ of $\Re^{n,n}$. Using equations 9.71 and 9.72 we have (where W_n and W_n^* are the pseudoscalars for $\mathcal{V}^n$ and $\mathcal{V}^{n*}$).

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[24] Since $\underline{K}^2 = 1$ we only need consider $E^{+}\left(\boldsymbol{e}_i, \boldsymbol{e}_j\right)$.

[25] The $\boldsymbol{\alpha} : \boldsymbol{E}$ notation simply says that indices of the scalar coefficients, α_{ij}, are balanced by the indices of the bivectors, E_{ij}. So that $\boldsymbol{\alpha} : \boldsymbol{E} = \sum_{i<j} \alpha_{ij} E_{ij}$ or $\boldsymbol{\gamma} : \boldsymbol{K} = \sum_i \gamma_i K_i$.

$$W_n = \boldsymbol{w}_1 \dots \boldsymbol{w}_n, \tag{9.133}$$

$$W_n^* = \boldsymbol{w}_1^* \dots \boldsymbol{w}_n^* \tag{9.134}$$

$$\underline{K}\left(W_n\right) = W_n, \tag{9.135}$$

$$\underline{K}\left(W_n^*\right) = \left(-1\right)^n W_n^*. \tag{9.136}$$

Thus when restricted to $\mathcal{V}^n$ or $\mathcal{V}^{n*}$, $\underline{K}$ is non-singular since the $\det\left(\underline{K}\right) \neq 0$.

Since each group element leaves $\mathcal{V}^n$ invariant

$$\underline{RK}\left(W_n\right) = \underline{R}W_n = RW_nR^\dagger = W_n, \tag{9.137}$$

we can write

$$\underline{R}\boldsymbol{w}_j = \sum_{k=1}^{n} \boldsymbol{w}_k \rho_{kj}. \tag{9.138}$$

Where using eq (9.35) we get

$$\rho_{ij} = 2\boldsymbol{w}_i^* \cdot \left(\underline{R}\boldsymbol{w}_j\right) = 2\left\langle \boldsymbol{w}_i^* R \boldsymbol{w}_j R^\dagger \right\rangle. \tag{9.139}$$

The rotations can be described on $\mathcal{V}^n$ without reference to $\Re^n$. For any member $\underline{R}$ of $GL\left(n, \Re\right)$ we have

$$\underline{R}W_n = W_n \det\nolimits_{\mathcal{V}^n}\left(\underline{R}\right), \tag{9.140}$$

where $\det_{\mathcal{V}^n}$ is the determinant of the linear transformation restricted to the $\mathcal{V}^n$ subspace of $\Re^{n,n}$ and not on the entire vector space. First note that

$$W_n^* \cdot W_n^\dagger = \left(w_1^* \dots w_n^*\right) \cdot \left(w_n \dots w_1\right) = \left\langle w_1^* \dots w_n^* w_n \dots w_1 \right\rangle = 2^{-n}. \tag{9.141}$$

Therefore,

$$\underline{R}W_n^\dagger = W_n^\dagger \det\nolimits_{\mathcal{V}^n}\left(\underline{R}^{-1}\right) \tag{9.142}$$

9.8 Endomorphisms of $\Re^n$

For a vector space an *endomorphism* is a linear mapping of the vector space onto itself. If the *endomorphism* has an inverse it is an *automorphism*. By studying the *endomorphisms* of $\Re_n$, the geometric algebra of $\Re^n$ we can also show in an alternative way that the mother algebra, $\Re_{n,n}$, is the appropriate arena for the study of linear transformations and Lie groups. First note that ($\simeq$ is the symbol for "is isomorphic to")

$$\mathcal{E}nd\left(\Re_n\right) \simeq \Re^{2^{2n}}, \tag{9.143}$$

since $End\left(\Re_n\right)$ is isomorphic to the algebra of all $2^n \times 2^n$ matrices.

For an arbitrary multivector $A \in \Re_n$, left and right multiplications by orthonormal basis vectors $\boldsymbol{e}_i$ determine endomorphisms of $\Re_n$ defined by

$$\underline{\boldsymbol{e}}_i : A \rightarrow \underline{\boldsymbol{e}}_i\left(A\right) \equiv \boldsymbol{e}_i A, \tag{9.144}$$

$$\underline{\bar{\boldsymbol{e}}}_i : A \rightarrow \underline{\bar{\boldsymbol{e}}}_i\left(A\right) \equiv \bar{A}\boldsymbol{e}_i \tag{9.145}$$

and the involution operator $\left(\bar{\bar{A}} = A\right)$ is defined by[26]

$$\overline{(AB)} = \bar{A}\bar{B} \tag{9.146}$$

$$\bar{\boldsymbol{e}}_i = -\boldsymbol{e}_i. \tag{9.147}$$

Thus for any multivector A we have

$$\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j\left(A\right) = \underline{\boldsymbol{e}}_i\left(\boldsymbol{e}_j A\right) = \boldsymbol{e}_i\boldsymbol{e}_j A, \tag{9.148}$$

$$\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j\left(A\right) = \underline{\boldsymbol{e}}_i\left(\bar{A}\boldsymbol{e}_j\right) = \boldsymbol{e}_i\bar{A}\boldsymbol{e}_j, \tag{9.149}$$

$$\underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i\left(A\right) = \underline{\bar{\boldsymbol{e}}}_j\left(\boldsymbol{e}_i A\right) = \overline{\boldsymbol{e}_i A}\boldsymbol{e}_j = -\boldsymbol{e}_i\bar{A}\boldsymbol{e}_j, \tag{9.150}$$

$$\underline{\bar{\boldsymbol{e}}}_i\underline{\bar{\boldsymbol{e}}}_j\left(A\right) = \underline{\bar{\boldsymbol{e}}}_i\left(\bar{A}\boldsymbol{e}_j\right) = \overline{\bar{A}\boldsymbol{e}_j}\boldsymbol{e}_i = -A\boldsymbol{e}_j\boldsymbol{e}_i. \tag{9.151}$$

Thus

$$\begin{aligned}
\left(\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i\right)\left(A\right) &= \left(\boldsymbol{e}_i\boldsymbol{e}_j + \boldsymbol{e}_j\boldsymbol{e}_i\right)A = 2\delta_{ij}A,\\
\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i &= 2\delta_{ij}, \qquad (9.152)\\
\left(\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i\right)\left(A\right) &= \boldsymbol{e}_i A\boldsymbol{e}_j - \boldsymbol{e}_i A\boldsymbol{e}_j = 0,\\
\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\boldsymbol{e}}_i &= 0, \qquad (9.153)\\
\left(\underline{\bar{\boldsymbol{e}}}_i\underline{\bar{\boldsymbol{e}}}_j + \underline{\bar{\boldsymbol{e}}}_j\underline{\bar{\boldsymbol{e}}}_i\right)\left(A\right) &= -A\left(\boldsymbol{e}_j\boldsymbol{e}_i + \boldsymbol{e}_i\boldsymbol{e}_j\right)A = -2\delta_{ij}A,\\
\underline{\boldsymbol{e}}_i\underline{\boldsymbol{e}}_j + \underline{\boldsymbol{e}}_j\underline{\boldsymbol{e}}_i &= -2\delta_{ij}. \qquad (9.154)
\end{aligned}$$

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[26] Since any grade r basis blade is of the form $\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$ with the j_k's in normal order then

$$\overline{\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}} = \bar{\boldsymbol{e}}_{j_1}\ldots\bar{\boldsymbol{e}}_{j_r} = \left(-1\right)^r\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$$

and the involute of any multivector, A, can be determined from equations (9.146) and (9.147). Likewise $\overline{\overline{\boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}}} = \boldsymbol{e}_{j_1}\ldots\boldsymbol{e}_{j_r}$.

Thus equations (9.152), (9.153), and (9.154) are isomorphic to equations (9.40), (9.42), and (9.41) which define the orthogonal basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$ of $\Re^{n,n}$. This establishes the isomorphism[27]

$$\Re_{n,n} \simeq \mathcal{E}nd\left(\Re_n\right). \tag{9.155}$$

The differences between $\Re_{n,n}$ and $\mathcal{E}nd\left(\Re_n\right)$ are that $\Re_{n,n}$ is the geometric algebra of a vector space with signature (n,n), basis $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$, and dimension 2^{2n}, while $\mathcal{E}nd\left(\Re_n\right)$ are the linear mappings of the geometric algebra $\Re_n$ on to itself with linear basis functions $\left\{\underline{\boldsymbol{e}}_i, \underline{\bar{\boldsymbol{e}}}_j\right\}$ that are isomorphic to $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$ and have the same anti-commutation relations (dot products) as $\{\boldsymbol{e}_i, \bar{\boldsymbol{e}}_j\}$.

Note that in defining $\underline{\bar{\boldsymbol{e}}}_i\left(A\right) = \bar{A}\boldsymbol{e}_i$, the involution, $\bar{A}$, is required to get the proper anti-commutation (dot product) relations that insure $\underline{\boldsymbol{e}}_i$ and $\underline{\bar{\boldsymbol{e}}}_j$ are orthogonal and that the signature of $\mathcal{E}nd\left(\Re_n\right)$ is (n,n).

Additionally, the composite operators

$$\begin{aligned}
\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i : A \rightarrow \underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i\left(A\right) &= \boldsymbol{e}_i\bar{A}\boldsymbol{e}_i && (9.156)\\
\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i\left(AB\right) &= \boldsymbol{e}_i\bar{A}\bar{B}\boldsymbol{e}_i\\
&= \boldsymbol{e}_i\bar{A}\boldsymbol{e}_i\boldsymbol{e}_i\bar{B}\boldsymbol{e}_i\\
&= \underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i\left(A\right)\underline{\boldsymbol{e}}_i\underline{\bar{\boldsymbol{e}}}_i\left(B\right) && (9.157)
\end{aligned}$$

preserve the geometric product and generate $\mathcal{A}ut\left(\Re_n\right)$, a subgroup of $\mathcal{E}nd\left(\Re_n\right)$.

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[27] $\dim\left(\Re_{n,n}\right) = 2^{2n}$.

Chapter 10

Classical Electromagnetic Theory

To formulate classical electromagnetic theory in terms of geometric algebra we will use the spacetime algebra with orthogonal basis vectors γ_0, γ_1, γ_2, and γ_3 and indexing convention that latin indices take on values 1 through 3 and greek indices values 0 through 3. The signature of the vector space is given by $\gamma_0^2 = -\gamma_i^2 = 1$ and the reciprocal basis is given by $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$. The relative basis vectors are given by $\vec{\sigma}_i = \gamma_i\gamma_0$. These bivectors are called relative vectors because their multiplication table is the same as for the geometric algebra of Euclidean 3-space (section 1.8.1) since $\vec{\sigma}_i^2 = 1$ and $\vec{\sigma}_i\vec{\sigma}_j = -\vec{\sigma}_j\vec{\sigma}_i$. Vector accents are used for relative vectors since it is usually used to denote vectors in ordinary (Euclidean) 3-space. We also have that $\vec{\sigma}_1\vec{\sigma}_2\vec{\sigma}_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = I$.

We now define relative electric and magnetic field vectors

$$\vec{E} \equiv E\gamma_0 = E^i\gamma_i\gamma_0 = E^i\vec{\sigma}_i \tag{10.1}$$

$$\vec{B} \equiv B\gamma_0 = B^i\gamma_i\gamma_0 = B^i\vec{\sigma}_i \tag{10.2}$$

We also have

$$\begin{aligned} \nabla\gamma_0 &= \left(\gamma^0\partial_0 + \gamma^i\partial_i\right)\gamma_0 && (10.3)\\ &= \partial_0 - \vec{\sigma}_i\partial_i && (10.4)\\ &= \partial_0 - \vec{\nabla} && (10.5) \end{aligned}$$

where we have defined

$$\vec{\nabla} \equiv \vec{\sigma}_i\partial_i \tag{10.6}$$

and also have

$$\gamma_0\nabla = \partial_0 + \gamma_0\gamma^i\partial_i = \partial_0 - \gamma_0\gamma_i\partial_i = \partial_0 + \gamma_i\gamma_0\partial_i = \partial_0 + \vec{\nabla} \tag{10.7}$$

Finally we have the 4-current $J = J^{\mu}\gamma_{\mu}$ where $J^0 = \rho$ then

$$J\gamma_0 = \rho + J^i\gamma_i\gamma_0 = \rho + J^i\vec{\sigma}_i = \rho + \vec{J}. \tag{10.8}$$

10.1 Maxwell Equations

The four vacuum Maxwell equations are then given by (for this section $\times$ is the 3-D vector product and remember in three dimensions $a \times b = -I(a \wedge b)$)

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{10.9}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{10.10}$$

$$-\vec{\nabla} \times \vec{E} = I\vec{\nabla} \wedge \vec{E} = \partial_0 \vec{B} \tag{10.11}$$

$$\vec{\nabla} \times \vec{B} = -I\vec{\nabla} \wedge \vec{B} = \partial_0 \vec{E} + \vec{J} \tag{10.12}$$

Now multiply equations 10.10 and 10.11 by I and $-I$ respectively and add 10.11 to 10.12 and 10.9 to 10.10 to get

$$\vec{\nabla} \cdot \left(\vec{E} + I\vec{B}\right) = \rho \tag{10.13}$$

$$\vec{\nabla} \wedge \left(\vec{E} + I\vec{B}\right) = -\partial_0 \left(\vec{E} + I\vec{B}\right) - \vec{J} \tag{10.14}$$

$$\vec{\nabla} \wedge \left(\vec{E} + I\vec{B}\right) + \partial_0 \left(\vec{E} + I\vec{B}\right) = -\vec{J} \tag{10.15}$$

Let

$$F = \vec{E} + I\vec{B} \tag{10.16}$$

so that

$$\vec{\nabla} \cdot F = \rho \tag{10.17}$$

$$\vec{\nabla} \wedge F + \partial_0 F = -\vec{J}. \tag{10.18}$$

Now remember ($\gamma_0 = \gamma^0$)

$$\begin{aligned}
\vec{\nabla} F &= \vec{\nabla} \wedge F + \vec{\nabla} \cdot F \\
\vec{\nabla} F &= -\partial_0 F + \rho - \vec{J} \\
\vec{\nabla} F + \partial_0 F &= \rho - \vec{J} \\
\gamma_0 \nabla F &= \rho - \vec{J} \\
\nabla F &= \rho\gamma_0 - J^i\gamma_0\gamma_i\gamma_0 \\
\nabla F &= \rho\gamma_0 + J^i\gamma_i
\end{aligned} \tag{10.19}$$

or

$$\boxed{\nabla F = J.} \tag{10.20}$$

Equation 10.20 contains all four vector Maxwell equations. Evaluating F gives

$$F = E^1\gamma_0\gamma_1 + E^2\gamma_0\gamma_2 + E^3\gamma_0\gamma_3 + B^1\gamma_2\gamma_3 - B^2\gamma_1\gamma_3 + B^3\gamma_1\gamma_2 \tag{10.21}$$

with the conserved quatities $\vec{E}^2 - \vec{B}^2$ and $\vec{E}\cdot\vec{B}$ since

$$F^2 = \vec{E}^2 - \vec{B}^2 + 2\left(\vec{E}\cdot\vec{B}\right)I = -FF^\dagger \tag{10.22}$$

and both scalars and pseudoscalars are unchange by spacetime rotations (Lorentz transformations).

The components of F as a tensor are given by $F^{\mu\nu} = (\gamma^\nu \wedge \gamma^\mu)\cdot F$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{bmatrix} \tag{10.23}$$

10.2 Relativity and Particles

Let $x(\lambda) = x^\nu(\lambda)\gamma_\nu$ be a parameterization of a particles world line (4-vector). The the proper time, τ, of the world line is the parameterization $\lambda(\tau)$ such that

$$\left(\frac{dx}{d\tau}\right)^2 = 1. \tag{10.24}$$

Since we are only considering physically accessible world lines where a particle velocity cannot equal or exceed the speed of light we have

$$\left(\frac{dx}{d\lambda}\right)^2 > 0. \tag{10.25}$$

The function $\lambda(\tau)$ can always be constructed as follows -

$$\frac{dx}{d\tau} = \frac{dx}{d\lambda}\frac{d\lambda}{d\tau} \tag{10.26}$$

$$\left(\frac{dx}{d\tau}\right)^2 = \left(\frac{dx}{d\lambda}\right)^2\left(\frac{d\lambda}{d\tau}\right)^2 = 1 \tag{10.27}$$

$$\frac{d\lambda}{d\tau} = \frac{1}{\sqrt{\left(\frac{dx}{d\lambda}\right)^2}} \tag{10.28}$$

$$\tau(\lambda) = \int_{\lambda_0}^{\lambda}\sqrt{\left(\frac{dx}{d\lambda'}\right)^2}\,d\lambda'. \tag{10.29}$$

From the form of equation 10.29 and equation 10.25 we know that $\tau(\lambda)$ is a monotonic function ($\tau(\lambda_2) > \tau(\lambda_1)\ \forall\ \lambda_2 > \lambda_1$) so that the inverse function $\lambda(\tau)$ exists $x(\tau) = x(\lambda(\tau))$.

Now define the 4-velocity v of a particle with world line $x(\tau)$ as

$$v \equiv \frac{dx}{d\tau} = \frac{dx^\nu}{d\tau}\gamma_\nu = v^\nu\gamma_\nu, \tag{10.30}$$

the relative 3-velocity (bivector) $\vec{\beta}$ as (remember that x^0 is the time coordinate in the local coordinate frame and τ is the time coordinate in the rest frame of the particle defined by $\vec{\beta} = 0$)

$$\vec{\beta} \equiv \frac{dx^i}{dx^0}\gamma_i\gamma_0 = \frac{dx^i}{dx^0}\vec{\sigma}_i, \tag{10.31}$$

and the relativistic γ factor as

$$\gamma \equiv \frac{dx^0}{d\tau}. \tag{10.32}$$

Thus we can write

$$v = \frac{dx^0}{d\tau}\left(\gamma_0 + \frac{dx^i}{dx^0}\gamma_i\right) \tag{10.33}$$

$$v = \gamma\left(\gamma_0 + \vec{\beta}\gamma_0\right) = \gamma\left(1 + \vec{\beta}\right)\gamma_0 \tag{10.34}$$

$$v^2 = \gamma^2\left(1 - \vec{\beta}^2\right) = 1 \tag{10.35}$$

$$\gamma^2 = \frac{1}{1 - \vec{\beta}^2} \tag{10.36}$$

$$\gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}}. \tag{10.37}$$

We also define the relativistic acceleration $\dot{v}$ by

$$\dot{v} \equiv \frac{dv}{d\tau}. \tag{10.38}$$

Note that since $v^2 = 1$ we have

$$\frac{d}{d\tau}(v \cdot v) = 2v \cdot \frac{dv}{d\tau} = 2v \cdot \dot{v} = 0. \tag{10.39}$$

10.3 Lorentz Force Law

From section 10.2 we have for the relativistic 4-velocity

$$\frac{dx}{d\tau} = \gamma\left(1 + \vec{\beta}\right)\gamma_0 \tag{10.40}$$

Then the covariant Lorentz force law is given by (q is the charge of the particle)

$$\begin{aligned} \frac{dp}{d\tau} = q\frac{dx}{d\tau} \cdot F = q\gamma\big(& \left(E^1\beta^1 + E^1\beta^1 + E^3\beta^3\right)\gamma_0 \\ & + \left(-B^2\beta^3 + B^3\beta^2 + E^1\right)\gamma_1 \\ & + \left(B^1\beta^3 - B^3\beta^1 + E^2\right)\gamma_2 \\ & + \left(-B^1\beta^2 + B^2\beta^1 + E^3\right)\gamma_3\big) \qquad (10.41) \\ = q\gamma & \left(\left(\vec{E} \cdot \vec{\beta}\right)\gamma_0 + \left(\left(\vec{E} + \vec{\beta} \times \vec{B}\right) \cdot \vec{\sigma}_i\right)\gamma_i\right) \qquad (10.42) \end{aligned}$$

where the component $q\gamma\left(\vec{E} \cdot \vec{\beta}\right)$ is the time derivative of the work done on the particle by the electric field in the specified coordinate frame.

10.4 Relativistic Field Transformations

A Lorentz transformation is described by a rotor $L = BR$ which is composed of a pure spatial rotation R and a relativistic boost rotation B. The general form of B is

$$B = \cosh\left(\frac{\alpha}{2}\right) - \sinh\left(\frac{\alpha}{2}\right)\hat{\beta} \tag{10.43}$$

where β is the magnitude of the relative 3-velocity (bivector) and $\hat{\beta}$ is a unit relative vector (bivector) in the direction of the 3-velocity. Then

$$\begin{aligned} \beta &= \tanh(\alpha) && (10.44)\\ \cosh(\alpha) = \gamma &= \frac{1}{\sqrt{1-\beta^2}} && (10.45)\\ \sinh(\alpha) &= \gamma\beta. && (10.46) \end{aligned}$$

If a boost is performed the relationship between the old basis vectors, γ_ν, and the new basis vectors, $\grave{\gamma}_\nu$, is

$$\grave{\gamma}_\nu = B\gamma_\nu B^\dagger. \tag{10.47}$$

If a boost is performed on the basis vectors we must have for the electomagnetic field bivector (remember $BB^\dagger = B^\dagger B = 1$)

$$\begin{aligned} F &= \grave{F}^{\mu\nu}\grave{\gamma}_\mu\grave{\gamma}_\nu = F^{\mu\nu}\gamma_\mu\gamma_\nu && (10.48)\\ F &= \grave{F}^{\mu\nu}B\gamma_\mu B^\dagger B\gamma_\nu B^\dagger = \grave{F}^{\mu\nu}B\gamma_\mu\gamma_\nu B^\dagger = F^{\mu\nu}\gamma_\mu\gamma_\nu && (10.49)\\ B^\dagger F B &= \grave{F}^{\mu\nu}\gamma_\mu\gamma_\nu = F^{\mu\nu}B^\dagger\gamma_\mu\gamma_\nu B. && (10.50) \end{aligned}$$

As an example consider the case of $\hat{\beta} = \gamma_1\gamma_0$ and

$$B = \cosh\left(\frac{\alpha}{2}\right) - \sinh\left(\frac{\alpha}{2}\right)\gamma_1\gamma_0 \tag{10.51}$$

where the velocity boost is along the 1-axis (x-axis). Then (after applying hyperbolic trig identities and double angle formulas)

$$\begin{aligned} B^\dagger F B = {} & E^1\gamma_0\gamma_1\\ & + \left(-B^3\sinh(\alpha) + E^2\cosh(\alpha)\right)\gamma_0\gamma_2\\ & + \left(B^3\cosh(\alpha) - E^2\sinh(\alpha)\right)\gamma_1\gamma_2\\ & + \left(B^2\sinh(\alpha) + E^3\cosh(\alpha)\right)\gamma_0\gamma_3\\ & + \left(-B^2\cosh(\alpha) - E^3\sinh(\alpha)\right)\gamma_1\gamma_3\\ & + B^1\gamma_2\gamma_3. \end{aligned} \tag{10.52}$$

Now using equations 10.44, 10.45, and 10.46 to get

$$\begin{aligned} B^{\dagger}FB = E^1\gamma_0\gamma_1 & \\ & + \gamma\left(-\beta B^3 + E^2\right)\gamma_0\gamma_2 \\ & + \gamma\left(-\beta E^2 + B^3\right)\gamma_1\gamma_2 \\ & + \gamma\left(\beta B^2 + E^3\right)\gamma_0\gamma_3 \\ & + \gamma\left(-\beta E^3 - B^2\right)\gamma_1\gamma_3 \\ & + B^1\gamma_2\gamma_3 \end{aligned} \tag{10.53}$$

Equating components of $B^{\dagger}FB$ gives

$$\begin{array}{rr} \grave{E}^1 = E^1 & \grave{B}^1 = B^1 \\ \grave{E}^2 = \gamma\left(E^2 - \beta B^3\right) & \grave{B}^2 = \gamma\left(B^2 + \beta E^3\right) \\ \grave{E}^3 = \gamma\left(E^3 + \beta B^2\right) & \grave{B}^3 = \gamma\left(B^3 - \beta E^2\right) \end{array} \tag{10.54}$$

for the transformed electomagnetic field components.

10.5 The Vector Potential

Starting with

$$\nabla F = \nabla\cdot F + \nabla\wedge F = J \tag{10.55}$$

and noting that $\nabla\cdot F$ and J are vectors and $\nabla\wedge F$ is a trivector we have

$$\nabla\cdot F = J \tag{10.56}$$

$$\nabla\wedge F = 0. \tag{10.57}$$

By equation 3.13 we can then write

$$F = \nabla\wedge A \tag{10.58}$$

$$\nabla\wedge F = \nabla\wedge\left(\nabla\wedge A\right) = 0. \tag{10.59}$$

The equation $\nabla\cdot F = J$ gives (use the fact that $\nabla G = \nabla\wedge G + \nabla\cdot G$ where G is a multivector field)

$$\begin{aligned} J &= \nabla\cdot\left(\nabla\wedge A\right) \\ &= \nabla\left(\nabla\wedge A\right) - \nabla\wedge\left(\nabla\wedge A\right) \\ &= \nabla\left(\nabla\wedge A\right) \\ &= \nabla\left(\nabla A - \nabla\cdot A\right) \\ &= \nabla^2 A - \nabla\left(\nabla\cdot A\right). \end{aligned} \tag{10.60}$$

We also note that if λ is a scalar field then

$$F = \nabla \wedge (A + \nabla\lambda) = \nabla \wedge A \tag{10.61}$$

since $\nabla \wedge (\nabla\lambda) = 0$. We have the freedom to add $\nabla\lambda$ to A without changing F. This is gauge invariance.

The Lorentz gauge is choosing $\nabla \cdot A = 0$, which is equivalent to $\nabla \cdot (\nabla\lambda) = -\nabla \cdot A$, so that the equation for the vector potential becomes

$$\nabla^2 A = J. \tag{10.62}$$

pdfl

10.6 Radiation from a Charged Particle

We now will calculate the electromagnetic fields radiated by a moving point charge. Let $x_p(\tau)$ be the 4-vector (space-time) trajectory of the point charge as a function of the charge's proper time, τ, and let x be the 4-vector observation point at which the fields are to be calculated. The 4-vector separation between the observer and the radiator is

$$X = x - x_p(\tau) = \left(x^0 - x_p^0(\tau)\right)\gamma_0 + (\vec{r} - \vec{r}_p)\,\gamma_0. \tag{10.63}$$

The critical relationship involving X is that it must lie on the light cone of $x_p(\tau)$ since we are dealing with electromagnetic radiation. That is X must be a null vector, $X^2 = 0$. $X(x,\tau)^2 = 0$ implies there is a functional relationship between τ and x so that we may write $\tau_r(x)$. $\tau_r(x)$ is the retarded time function. We use retarded time to mean the smaller solution (earlier time) of $X(x,\tau)^2 = 0$ since the equation must always have two solutions. The larger solution would be the advanced time and would correspond to signal travelling faster than light. In all these calculations $c = 1$ and the relative 3-velocity (bivector) of the moving charge is $\vec{\beta}_p$ (section 10.2).

The 4-velocity and acceleration of the point charge are given by

$$v_p = \frac{dx_p}{d\tau} = \gamma_p\left(1 + \vec{\beta}_p\right)\gamma_0 \tag{10.64}$$

$$\dot{v}_p = \frac{dv_p}{d\tau} \tag{10.65}$$

The vector potential of a moving point charge of charge q is given by the Liènard-Wiechart potential[1]

$$\varphi\left(\vec{r},t\right) = \frac{q}{4\pi}\left(\frac{1}{\left(1-\vec{n}\cdot\vec{\beta}_p\right)\left|\vec{r}-\vec{r_s}\right|}\right)_{\tau_r(x)} \tag{10.66}$$

$$\vec{A}\left(\vec{r},t\right) = \vec{\beta}_p\left(\tau_r\left(x\right)\right)\varphi\left(\vec{r},t\right) \tag{10.67}$$

where

$$\vec{n} = \frac{\vec{r}-\vec{r_p}}{\left|\vec{r}-\vec{r_p}\right|} \tag{10.68}$$

or in terms of 4-vectors

$$A = \left(\varphi\left(\vec{r},t\right)+\vec{A}\left(\vec{r},t\right)\right)\gamma_0 = \frac{q}{4\pi}\frac{v_p}{\left|X\cdot v_p\right|} \tag{10.69}$$

and we must calculate $\nabla\wedge A$. In the following derivation whenever we write X we mean $X\left(x,\tau_r\left(x\right)\right)$. X is always evaluated at the retarded time.

To start consider that if we know $\tau_r\left(x\right)$ implicitly defined by $X\left(x,\tau_r\right)^2 = 0$ then the function $X\left(x,\tau_r\left(x\right)\right)^2 \equiv 0$ (identically equal to zero). Thus $\nabla\left(X\left(x,\tau_r\left(x\right)\right)^2\right) \equiv 0$. First note that (by the symmetry of $g_{\mu\nu}$)

$$\begin{aligned}\nabla\left(X^2\right) &= \gamma^\eta\partial_\eta\left(X^\mu X^\nu g_{\mu\nu}\right)\\ &= \gamma^\eta\left(\left(\partial_\eta X^\mu\right)X^\nu + X^\mu\partial_\eta X^\nu\right)g_{\mu\nu}\\ &= 2\gamma^\eta\left(\partial_\eta X^\mu\right)X^\nu g_{\mu\nu}\\ &= 2\gamma^\eta\left(\partial_\eta X\right)\cdot X.\end{aligned} \tag{10.70}$$

Thus $\gamma^\eta\left(\partial_\eta X\right)\cdot X = 0$ is equivalent to $\nabla\left(X^2\right) = 0$. But

$$\begin{aligned}\gamma^\eta\left(\partial_\eta X\right)\cdot X &= \gamma^\eta\left(\partial_\eta x\right)\cdot X - \gamma^\eta\left(\partial_\eta x_p\right)\cdot X\\ &= \gamma^\eta\left(\frac{\partial x^\mu}{\partial x^\eta}\gamma_\mu\right)\cdot X^\nu\gamma_\nu - \gamma^\eta\left(\frac{\partial x_p^\mu}{\partial x^\eta}\gamma_\mu\right)\cdot X\\ &= \gamma^\eta\delta_\eta^\mu X^\nu g_{\mu\nu} - \gamma^\eta\left(\frac{\partial x_p^\mu}{\partial\tau_r}\frac{\partial\tau_r}{\partial x^\eta}\gamma_\mu\right)\cdot X\\ &= g_{\mu\nu}\gamma^\mu X^\nu - \gamma^\eta\frac{\partial\tau_r}{\partial x^\eta}\left(\frac{\partial x_p^\mu}{\partial\tau_r}\gamma_\mu\right)\cdot X\\ &= X - \nabla\tau_r\left(v_p\cdot X\right) = 0\end{aligned} \tag{10.71}$$

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[1] en.wikipedia.org/wiki/Liènard-Wiechert_potential

or

$$\nabla \tau_r = \frac{X}{X \cdot v_p}. \tag{10.72}$$

Another simple relation we need is

$$\begin{aligned}
\nabla v_p &= \gamma^\mu \frac{\partial v_p^\nu}{\partial x^\mu} \gamma_\nu \\
&= \gamma^\mu \frac{\partial v_p^\nu}{\partial \tau_r} \frac{\partial \tau_r}{\partial x^\mu} \gamma_\nu \\
&= (\nabla \tau_r) \dot{v}_p \\
&= \frac{X \dot{v}_p}{X \cdot v_p}. \qquad (10.73)
\end{aligned}$$

The last quantity we need is $\nabla (X \cdot v)$

$$\begin{aligned}
\nabla (X \cdot v) &= \gamma^\mu \partial_\mu \left(\left(x^\nu - x_p^\nu \right) v_p^\eta g_{\nu\eta} \right) \\
&= \gamma^\mu \left(\left(\frac{\partial x^\nu}{\partial x^\mu} - \frac{\partial x_p^\nu}{\partial \tau_r} \frac{\partial \tau_r}{\partial x^\mu} \right) v_p^\eta g_{\nu\eta} + X^\nu \frac{\partial v_p^\eta}{\partial \tau_r} \frac{\partial \tau_r}{\partial x^\mu} g_{\mu\nu} \right) \\
&= \gamma^\nu g_{\nu\eta} v_p^\eta - \gamma^\mu \frac{\partial \tau_r}{\partial x^\mu} \frac{\partial x_p^\nu}{\partial \tau_r} g_{\nu\eta} v_p^\eta + \gamma^\mu \frac{\partial \tau_r}{\partial x^\mu} X^\nu g_{\mu\nu} \frac{\partial v_p^\eta}{\partial \tau_r} \\
&= v_p - \nabla \tau_r \left(v_p \cdot v_p - X \cdot \dot{v}_p \right) \\
&= v_p - \frac{X}{v_p \cdot X} \left(v_p \cdot v_p - X \cdot \dot{v}_p \right) \\
&= v_p + \frac{X \left(X \cdot \dot{v}_p - 1 \right)}{X \cdot v_p} \qquad (10.74)
\end{aligned}$$

Now we can calculate ∇A

$$\begin{aligned}
\nabla A &= \frac{q}{4\pi}\left(\frac{\nabla v_p}{X\cdot v_p} - \frac{1}{\left(X\cdot v_p\right)^2}\nabla\left(X\cdot v_p\right)v_p\right) \\
&= \frac{q}{4\pi}\left(\frac{X\dot{v}_p}{\left(X\cdot v_p\right)^2} - \frac{1}{\left(X\cdot v_p\right)^2}\left(v_p + \frac{X\left(X\cdot\dot{v}_p - 1\right)}{X\cdot v_p}\right)v_p\right) \\
&= \frac{q}{4\pi\left(X\cdot v_p\right)^2}\left(X\dot{v}_p - 1 - \frac{Xv_p\left(X\cdot\dot{v}_p - 1\right)}{X\cdot v_p}\right) \\
&= \frac{q}{4\pi\left(X\cdot v_p\right)^2}\left(X\wedge\dot{v}_p + \frac{X\wedge v_p - \left(X\cdot\dot{v}_p\right)X\wedge v_p}{X\cdot v_p}\right) \\
&= \frac{q}{4\pi\left(X\cdot v_p\right)^3}\left(X\wedge v_p + \left(X\wedge\dot{v}_p\right)\left(X\cdot v_p\right) - \left(X\cdot\dot{v}_p\right)\left(X\wedge v_p\right)\right) \qquad (10.75)
\end{aligned}$$

Note that equation 10.75 is a pure bivector so that $\nabla\cdot A = 0$ and $\nabla\wedge A = \nabla A$ so that $F = \nabla A$. Now expand $\left(X\wedge\dot{v}_p\right)\left(X\cdot v_p\right) - \left(X\cdot\dot{v}_p\right)\left(X\wedge v_p\right)$ using the definitons of $\wedge$ and $\cdot$ for vectors to get

$$F = \frac{q}{4\pi\left(X\cdot v_p\right)^3}\left(X\wedge v_p + \frac{1}{2}X\left(\dot{v}_p\wedge v_p\right)X\right). \qquad (10.76)$$

The first term in equation 10.76 falls of as $\dfrac{1}{r^2}$ and is the static field term. The second term falls off as $\dfrac{1}{r}$ and is the radiation field term so that

$$F_{\text{rad}} = \frac{qX\left(\dot{v}_p\wedge v_p\right)X}{8\pi\left(X\cdot v_p\right)^3}. \qquad (10.77)$$

Appendix A

Further Properties of the Geometric Product

For two vectors a and b we can revise the order of the geometric product via the formula

$$ba = 2a \cdot b - ab. \tag{A.1}$$

By repeated applications of equation A.1 we can relate the geometric product $aa_1 \dots a_r$ to the geometric product $a_1 \dots a_r a$ -

$$\begin{aligned} aa_1 \dots a_r &= 2\left(a \cdot a_1\right) a_2 \dots a_r - a_1 a a_2 \dots a_r \\ &= 2\left(a \cdot a_1\right) a_2 \dots a_r - 2\left(a \cdot a_2\right) a_1 a_3 \dots a_r + a_1 a_2 a a_3 \dots a_r \\ &= 2 \sum_{k=1}^{r} \left(-1\right)^{k+1} \left(a \cdot a_k\right) a_1 \dots \breve{a}_k \dots a_r + \left(-1\right)^r a_1 \dots a_r a. \end{aligned} \tag{A.2}$$

Thus

$$\begin{aligned} \frac{1}{2}\left(aa_1 \dots a_r - \left(-1\right)^r a_1 \dots a_r a\right) &= \sum_{k=1}^{r} \left(-1\right)^{k+1} \left(a \cdot a_k\right) a_1 \dots \breve{a}_k \dots a_r \\ a \cdot \left(a_1 \dots a_r\right) &= \sum_{k=1}^{r} \left(-1\right)^{k+1} \left(a \cdot a_k\right) a_1 \dots \breve{a}_k \dots a_r. \end{aligned} \tag{A.3}$$

Now consider the expression $a \cdot (a_1 \wedge \dots \wedge a_r)$ and how it can be reduced. Since $a_1 \wedge \dots \wedge a_r$ only contains grades r, $r-2$, …, we have

$$\begin{aligned} a\cdot(a_1\dots a_r) &= \frac{1}{2}\left(aa_1\dots a_r - (-1)^r a_1\dots a_r a\right) \\ &= a\cdot\langle a_1\dots a_r\rangle_r + a\cdot\langle a_1\dots a_r\rangle_{r-2} + \cdots \end{aligned} \tag{A.4}$$

The term we need is the $r-1$ grade part so that

$$\begin{aligned} a\cdot(a_1\wedge\dots\wedge a_r) &= \frac{1}{2}\langle aa_1\dots a_r - (-1)^r a_1\dots a_r a\rangle_{r-1} \\ &= \sum_{k=1}^{r}(-1)^{k+1}(a\cdot a_k)\langle a_1\dots\breve{a}_k\dots a_r\rangle_{r-1} \\ &= \sum_{k=1}^{r}(-1)^{k+1}(a\cdot a_k)(a_1\wedge\dots\wedge\breve{a}_k\wedge\dots\wedge a_r) \\ &= \sum_{k=1}^{r}(-1)^{k-1}(a\cdot a_k)(a_1\wedge\dots\wedge\breve{a}_k\wedge\dots\wedge a_r). \end{aligned} \tag{A.5}$$

If A_r, B_s, and C_t are pure grade multivectors of grade r, s, and t we have

$$A_r\cdot(B_s\cdot C_t) = (A_r\wedge B_s)\cdot C_t \;\forall\; r+s\le t \text{ and } r,s>0 \tag{A.6}$$

$$A_r\cdot(B_s\cdot C_t) = (A_r\cdot B_s)\cdot C_t \;\forall\; r+t\le s. \tag{A.7}$$

To prove equation A.6

$$\begin{aligned} A_r\cdot(B_s\cdot C_t) &= A_r\cdot\langle B_sC_t\rangle_{t-s} \\ &= \left\langle A_r\langle B_sC_t\rangle_{t-s}\right\rangle_{t-(r+s)} \\ &= \left\langle \langle A_rB_sC_t\rangle_{t-(r+s)} + \text{higher grades}\right\rangle_{t-(r+s)} \\ &= \langle A_rB_sC_t\rangle_{t-(r+s)} \end{aligned} \tag{A.8}$$

and

$$\begin{aligned} (A_r\wedge B_s)\cdot C_t &= \langle A_rB_s\rangle_{r+s}\cdot C_t \\ &= \left\langle\langle A_rB_s\rangle_{r+s}C_t\right\rangle_{t-(r+s)} \\ &= \left\langle \langle A_rB_sC_t\rangle_{t-(r+s)} + \text{higher grades}\right\rangle_{t-(r+s)} \\ &= \langle A_rB_sC_t\rangle_{t-(r+s)}. \end{aligned} \tag{A.9}$$

To prove equation A.7

$$\begin{aligned}
A_r \cdot (B_s \cdot C_t) &= A_r \cdot \langle B_s C_t \rangle_{s-t} \\
&= \left\langle A_r \langle B_s C_t \rangle_{s-t} \right\rangle_{s-(r+t)} \\
&= \left\langle \langle A_r B_s C_t \rangle_{s-(r+t)} + \text{higher grades} \right\rangle_{s-(r+t)} \\
&= \langle A_r B_s C_t \rangle_{s-(r+t)} \qquad \text{(A.10)}
\end{aligned}$$

and

$$\begin{aligned}
(A_r \cdot B_s) \cdot C_t &= \langle A_r B_s \rangle_{s-r} \cdot C_t \\
&= \left\langle \langle A_r B_s \rangle_{s-r} C_t \right\rangle_{s-(r+t)} \\
&= \left\langle \langle A_r B_s C_t \rangle_{s-(r+t)} + \text{higher grades} \right\rangle_{s-(r+t)} \\
&= \langle A_r B_s C_t \rangle_{s-(r+t)} . \qquad \text{(A.11)}
\end{aligned}$$

If A_n is the pseudoscalar of an n-dimensional subspace of the vector space and $B_{\overline{s}}$ is a blade (Hestenes puts a line over the subscript or superscript to indicate that the pure grade multivector is a blade or as he says is simple) then

$$P_{A_n}\left(B_{\overline{s}}\right) = B_{\overline{s}} \text{ iff } B_{\overline{s}} A_n = B_{\overline{s}} \cdot A_n. \qquad \text{(A.12)}$$

First remember that $P_{A_n}(B) = (B \cdot A_n) A_n^{-1}$. Since both A_n and $B_{\overline{s}}$ are blades they both can be written as the geometric product of orthogonal vectors. Let $A_n = u_1^A \dots u_n^A$ and $B_{\overline{s}} = u_1^A \dots u_r^A u_{r+1}^B \dots u_s^B$. The expression for $B_{\overline{s}}$ indicate that it could contain r of the same basis vectors of A_n. Also remember that

$$A_n^{-1} = \frac{u_n^A}{\left(u_n^A\right)^2} \cdots \frac{u_1^A}{\left(u_1^A\right)^2} \qquad \text{(A.13)}$$

and evaluate

$$B_{\overline{s}} A_n = u_1^A \dots u_r^A u_{r+1}^B \dots u_s^B u_1^A \dots u_n^A. \qquad \text{(A.14)}$$

Since $r \leq n$ the number of orthogonal vectors in equation A.14 and hence the grade of $B_{\overline{s}} A_n$ is $n + s - 2r \geq 0$. If $s \leq n$ and subspace defined by $B_{\overline{s}}$ is contained by the subspace defined by A_n then the grade of

$$B_{\overline{s}} A_n = u_1^A \dots u_s^A u_1^A \dots u_n^A \qquad \text{(A.15)}$$

is $n-s$ and $B_{\overline{s}}A_n = B_{\overline{s}}\cdot A_n$. Thus

$$\begin{aligned}(B_{\overline{s}}\cdot A_n)\,A_n^{-1} &= \left(u_1^A\dots u_s^A u_1^A\dots u_n^A\right)\frac{u_n^A}{\left(u_n^A\right)^2}\cdots\frac{u_1^A}{\left(u_1^A\right)^2}\\ &= u_1^A\dots u_s^A\\ &= B_{\overline{s}}.\end{aligned}\tag{A.16}$$

Now prove that for the integers $1\le i_1<\cdots<i_r\le n$ and $1\le j_1<\cdots<j_r\le n$ we have

$$\left(u^{i_r}\wedge\dots\wedge u^{i_1}\right)\cdot\left(u_{j_1}\wedge\dots\wedge u_{j_r}\right) = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\dots\delta^{i_r}_{j_r}.\tag{A.17}$$

Start by letting $A_n = u_{j_1}\wedge u_{j_2}\wedge\dots\wedge u_{j_n}$ be a pseudoscalar for the vector space, then from equation 1.61 we have

$$u^{i_l} = (-1)^{i_l-1}\,u_1\wedge\dots\wedge\breve{u}_{i_l}\wedge\dots\wedge u_nA_n^{-1}.\tag{A.18}$$

Since u^{i_l} is in the space defined by A_n we have $u^{i_l}A_n = u^{i_l}\cdot A_n$. Now consider the progression

$$u^{i_1}A_n = u^{i_1}\cdot A_n = (-1)^{i_1-1}\,u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge u_n\tag{A.19}$$

$$\begin{aligned}\left(u^{i_2}\wedge u^{i_1}\right)A_n &= \left(u^{i_2}\wedge u^{i_1}\right)\cdot A_n\\ &= u^{i_2}\cdot\left(u^{i_1}\cdot A_n\right)\text{ from equation A.6} &&\text{(A.20)}\\ &= (-1)^{i_1-1}\,u^{i_2}\cdot\left(u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge u_n\right) &&\text{(A.21)}\\ &= (-1)^{i_1-1}\sum_{k\ne i_1}^{r}(-1)^{k-1}\,u^{i_2}\cdot u_k\left(u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge\breve{u}_k\wedge\dots\wedge u_n\right) &&\text{(A.22)}\\ &= (-1)^{i_1-1}(-1)^{i_2-2}\,u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge\breve{u}_{i_2}\wedge\dots\wedge u_n. &&\text{(A.23)}\end{aligned}$$

Equation A.23 is obtained by applying equation A.5 to equation A.22. The reason that in equation A.23 the second power of -1 is i_2-2 and not i_2-1 is that in equation A.21 u_{i_1} has been removed from the wedge product. Since

$$\begin{aligned}\left(u^{i_k}\wedge\dots\wedge u^{i_1}\right)A_r &= \left(u^{i_k}\wedge\dots\wedge u^{i_1}\right)\cdot A_r\\ &= \left(u^{i_k}\wedge\left(u^{i_{k-1}}\dots\wedge u^{i_1}\right)\right)\cdot A_r\\ &= u^{i_k}\cdot\left(\left(u^{i_{k-1}}\dots\wedge u^{i_1}\right)\cdot A_r\right)\end{aligned}\tag{A.24}$$

so by induction

$$\begin{aligned}\left(u^{i_r}\wedge\dots\wedge u^{i_1}\right)A_r &= (-1)^{\sum_{l=1}^{r}(i_l-l)}\,u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge\breve{u}_{i_r}\wedge\dots\wedge u_n &&\text{(A.25)}\\ u^{i_r}\wedge\dots\wedge u^{i_1} &= (-1)^{\sum_{l=1}^{r}(i_l-l)}\left(u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge\breve{u}_{i_r}\wedge\dots\wedge u_n\right)A_n^{-1} &&\text{(A.26)}\\ u^{i_r}\wedge\dots\wedge u^{i_1} &= (-1)^{-\frac{r(r+1)}{2}}(-1)^{\sum_{l=1}^{r}i_l}\left(u_1\wedge\dots\wedge\breve{u}_{i_1}\wedge\dots\wedge\breve{u}_{i_r}\wedge\dots\wedge u_n\right)A_n^{-1}. &&\text{(A.27)}\end{aligned}$$

Now let the indices $\{i_{r+1},\ldots,i_n\}$ be the complement of the indices $\{i_1,\ldots,i_r\}$ in the set $\{1,\ldots,n\}$ with the condition that $0 < i_{r+1} < \cdots < i_n \le n$. Then we may write equation A.27 as

$$u^{i_r}\wedge\ldots\wedge u^{i_1} = (-1)^{-\frac{r(r+1)}{2}}\,(-1)^{\sum_{l=1}^{r} i_l}\left(u_{i_{r+1}}\wedge\ldots\wedge u_{i_n}\right)A_n^{-1}. \tag{A.28}$$

Then

$$\begin{aligned}
\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\right)\cdot\left(u^{j_1}\wedge\ldots\wedge u^{j_r}\right) &= (-1)^{\frac{r(r-1)}{2}}\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\right)\cdot\left(u^{j_r}\wedge\ldots\wedge u^{j_1}\right)\\
&= (-1)^{-r}\,(-1)^{\sum_{l=1}^{r} j_l}\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\right)\cdot\left(\left(u_{j_{r+1}}\wedge\ldots\wedge u_{j_n}\right)A_n^{-1}\right)\\
&= (-1)^{\sum_{l=1}^{r}(j_l-1)}\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\right)\cdot\left(\left(u_{j_{r+1}}\wedge\ldots\wedge u_{j_n}\right)\cdot A_n^{-1}\right)\\
&= (-1)^{\sum_{l=1}^{r}(j_l-1)}\left(\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\right)\wedge\left(u_{j_{r+1}}\wedge\ldots\wedge u_{j_n}\right)\right)\cdot A_n^{-1}\\
&= (-1)^{\sum_{l=1}^{r}(j_l-1)}\,u_{i_r}\wedge\ldots\wedge u_{i_1}\wedge u_{j_{r+1}}\wedge\ldots\wedge u_{j_n}A_n^{-1} \qquad \text{(A.29)}\\
&= \delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\ldots\delta_{i_r}^{j_r}. \qquad \text{(A.30)}
\end{aligned}$$

The step from equation A.29 to equation A.30 requires a detailed explanation. First the r.h.s. of equation A.29 is zero unless there are no repetitions in the index list $[i_r,\ldots,i_1,j_{r+1},\ldots,j_n]$, but there are no repetitions if $\delta_{i_1}^{j_1}\delta_{i_2}^{j_2}\ldots\delta_{i_r}^{j_r}\neq 0$. Because both the i_l's and the j_l's are ordered we must match $i_l = j_l\ \forall\ 1\le l\le r$ so that in the Kronecker delta's the index subscripts match. Also note that in equation A.29 that the subscript list $[i_r,\ldots,i_1,j_{r+1},\ldots,j_n]$ is not in ascending order so that if not zero $\left(u_{i_r}\wedge\ldots\wedge u_{i_1}\wedge u_{j_{r+1}}\wedge\ldots\wedge u_{j_n}\right)A_n^{-1} = \pm 1$. The factor $(-1)^{\sum_{l=1}^{r}(j_l-1)}$ is required to order the subscript list. For the nonzero products we have

$$\left(u_{j_r}\wedge\ldots\wedge u_{j_1}\right)\cdot\left(u^{j_1}\wedge\ldots\wedge u^{j_r}\right) = (-1)^{j_r-1}\,u_{j_r}\wedge\ldots\wedge(-1)^{j_1-1}\,u_{j_1}\wedge u_{i_{r+1}}\wedge\ldots\wedge u_{j_n}A_n^{-1}. \tag{A.31}$$

Now remember that $j_l\ l\le l\le r$ is the absolute position index of a basis vector in the pseudoscalar product. To move u_{j_r} in equation A.31 to its correct position in the pseudoscalar product requires $j_r - 1$ transpositions and hence the factor $(-1)^{j_r-1}$. Repeating the proceedure for basis vectors $u_{j_{r-1}}$ to u_{j_1} reduces the r.h.s. of equation A.31 to 1.

Appendix B

BAC-CAB Formulas

Using the python geometric algebra module GA[1] several formulas containing the dot and wedge products can be reduced. Let a, b, c, d, and e be vectors, then we have

$$\begin{aligned}
a\cdot(bc) &= (b\cdot c)\,a-(a\cdot c)\,b+(a\cdot b)\,c && \text{(B.1)}\\
a\cdot(b\wedge c) &= (a\cdot b)\,c-(a\cdot c)\,b && \text{(B.2)}\\
a\cdot(b\wedge c\wedge d) &= (a\cdot d)\,(b\wedge c)-(a\cdot c)\,(b\wedge d)\\
&\quad+(a\cdot b)\,(c\wedge d) && \text{(B.3)}\\
a\cdot(b\wedge c\wedge d\wedge e) &= -\,(a\cdot e)\,(b\wedge c\wedge d)+(a\cdot d)\,(b\wedge c\wedge e)\\
&\quad-(a\cdot c)\,(b\wedge d\wedge e)+(a\cdot b)\,(c\wedge d\wedge e) && \text{(B.4)}\\
(a\cdot(b\wedge c))\cdot(d\wedge e) &= ((a\cdot c)\,(b\cdot e)-(a\cdot b)\,(c\cdot e))\,d\\
&\quad+((a\cdot b)\,(c\cdot d)-(a\cdot c)\,(b\cdot d))\,e. && \text{(B.5)}
\end{aligned}$$

If in equation B.2 the vector b is replace by a vector differential operator such as ∇, ∂, or D (we will use D as an example) it can be rewritten as

$$\begin{aligned}
(a\cdot D)\,c &= a\cdot(D\wedge c)+(a\cdot\dot{c})\,\dot{D}\\
&= a\cdot(D\wedge c)+\dot{D}\,(a\cdot\dot{c})\\
&= a\cdot(D\wedge c)+\dot{D}\,(\dot{c}\cdot a) && \text{(B.6)}
\end{aligned}$$

Cyclic reduction formulas are

$$a\cdot(b\wedge c)+c\cdot(a\wedge b)+b\cdot(c\wedge a)=0 \qquad \text{(B.7)}$$

---

[1] Alan Macdonald website `http://faculty.luther.edu/~macdonal/vagc/index.html`

$$a\,(b\wedge c) - b\,(a\wedge c) + c\,(a\wedge b) = 3a\wedge b\wedge c \tag{B.8}$$

$$\begin{aligned} a\,(b\wedge c\wedge d) - b\,(a\wedge c\wedge d) + c\,(a\wedge b\wedge d) - d\,(a\wedge b\wedge c) = \\ 4a\wedge b\wedge c\wedge d \end{aligned} \tag{B.9}$$

Basis blade reduction formula

$$\begin{aligned} (a\wedge b)\cdot(c\wedge d) &= ((a\wedge b)\cdot c)\cdot d \\ &= (a\cdot d)\,(b\cdot c) - (a\cdot c)\,(b\cdot d) \end{aligned} \tag{B.10}$$

But we also have that

$$((a\wedge b)\cdot c)\cdot d = (a\cdot d)\,(b\cdot c) - (a\cdot c)\,(b\cdot d) \tag{B.11}$$

which gives the same results as equation B.10. Since for any bivector blade $B = a\wedge c$ we have

$$(B\cdot c)\cdot d = B\cdot(c\wedge d)\,. \tag{B.12}$$

By linearity equation B.12 is also true for any bivector B since B is a superposition of bivector blades.

Finally one formula for reducing the commutator product of two bivectors

$$\begin{aligned} (a\wedge b)\times(c\wedge d) &= (a\cdot d)\,b\wedge c - (a\cdot c)\,b\wedge d \\ &\quad + (b\cdot c)\,a\wedge d - (b\cdot d)\,a\wedge c \end{aligned} \tag{B.13}$$

Appendix C

Reduction Rules for Scalar Projections of Multivectors

If A is a general multivector define the even, A^{+}, and odd, A^{-}, components of A by

$$A^{+} = \left\langle A\right\rangle_0 + \left\langle A\right\rangle_2 + \cdots \tag{C.1}$$

$$A^{-} = \left\langle A\right\rangle_1 + \left\langle A\right\rangle_3 + \cdots . \tag{C.2}$$

Then the following rules are used for the reduction of the scalar projections of multivector expressions.

Reduction Rule 0 (RR0) $\left\langle A^{-}\right\rangle = 0$.

Reduction Rule 1 (RR1) Let $A = a_1 \cdots a_r$, then if r is even $A = A^{+}$ and if r is odd $A = A^{-}$

> If $r = 2$ we know that $A = A^{+}$. Assume RR1 is true for r even and multiply the result by a vector a then
>
> $$\begin{aligned} Aa &= \left(\left\langle A\right\rangle_0 + \left\langle A\right\rangle_2 + \cdots\right) a \\ &= \left\langle A\right\rangle_0 a + \left\langle\left\langle A\right\rangle_2 a\right\rangle_1 + \left\langle\left\langle A\right\rangle_2 a\right\rangle_3 + \cdots . \end{aligned} \tag{C.3}$$
>
> Now assume that $A = A^{-}$ and multiply by a to get
>
> $$\begin{aligned} Aa &= \left(\left\langle A\right\rangle_1 + \left\langle A\right\rangle_3 + \cdots\right) a \\ &= \left\langle\left\langle A\right\rangle_1 a\right\rangle_0 + \left\langle\left\langle A\right\rangle_1 a\right\rangle_2 + \left\langle\left\langle A\right\rangle_3 a\right\rangle_2 + \left\langle\left\langle A\right\rangle_3 a\right\rangle_4 + \cdots . \end{aligned} \tag{C.4}$$

Reduction Rule 2 (RR2) For any two general multivectors A and B

$$\begin{aligned} A^+B^+ &= \left(A^+B^+\right)^+ && \text{(C.5)}\\ A^-B^- &= \left(A^-B^-\right)^+ && \text{(C.6)}\\ A^-B^+ &= \left(A^-B^+\right)^- && \text{(C.7)}\\ A^+B^- &= \left(A^+B^-\right)^- . && \text{(C.8)} \end{aligned}$$

$A^+ = \sum_i \left\langle A \right\rangle_{2j}$ and $A^- = \sum_i \left\langle A \right\rangle_{2j-1}$ and likewise for B^+ and B^-. Thus the general term in A^+B^+ if of the form

$$\begin{aligned} \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} = {}& \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} \right\rangle_{|2(i-j)|} + \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} \right\rangle_{|2(i-j)|+2} + \cdots \\ & + \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} \right\rangle_{|2(i+j)|} \end{aligned} \tag{C.9}$$

which has only even grades. The general term in A^+B^- if of the form

$$\begin{aligned} \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j-1} = {}& \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j-1} \right\rangle_{|2(i-j)+1|} + \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} \right\rangle_{|2(i-j)|+3} + \cdots \\ & + \left\langle \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j} \right\rangle_{|2(i+j)-1|} \end{aligned} \tag{C.10}$$

which has only odd grades. The general term in A^-B^- if of the form

$$\begin{aligned} \left\langle A \right\rangle_{2i} \left\langle B \right\rangle_{2j-1} = {}& \left\langle \left\langle A \right\rangle_{2i-1} \left\langle B \right\rangle_{2j-1} \right\rangle_{|2(i-j)|} + \left\langle \left\langle A \right\rangle_{2i-1} \left\langle B \right\rangle_{2j-1} \right\rangle_{|2(i-j)|+2} + \cdots \\ & + \left\langle \left\langle A \right\rangle_{2i-1} \left\langle B \right\rangle_{2j-1} \right\rangle_{|2(i+j)-2|} \end{aligned} \tag{C.11}$$

which has only even grades.

Reduction Rule 3 (RR3) If B_s is an s-grade multivector $\left\langle \left\langle A \right\rangle_r B_s \right\rangle = \delta_{rs} \left\langle AB_s \right\rangle$

The lowest grade in $\left\langle A \right\rangle_r B_s$ is $|r-s|$ so if $r \neq s$ there is no zero grade.

Reduction Rule 4 (RR4) If A_r is an r-grade multivector $\left\langle A_rB \right\rangle = A_r * \left\langle B \right\rangle_r$

We have from RR3 that $\left\langle A_rB \right\rangle = \left\langle A_r \left\langle B \right\rangle_r \right\rangle = A_r * \left\langle B \right\rangle_r$.

Reduction Rule 5 (RR5) $\left\langle A_1 \ldots A_k \right\rangle = \left\langle A_jA_{j+1} \ldots A_kA_1 \ldots A_{j-1} \right\rangle$

Using equation 6.17 we have

$$\begin{aligned}\langle (A_1A_2\dots A_{k-1})\, A_k\rangle &= \left\langle (A_1A_2\dots A_{k-1})^\dagger\, A_k^\dagger \right\rangle \\ &= \langle A_k\,(A_1A_2\dots A_{k-1})\rangle \\ &= \left\langle (A_k\,(A_1A_2\dots A_{k-1}))^\dagger \right\rangle \\ &= \langle A_k\,(A_1A_2\dots A_{k-1})\rangle\,. \end{aligned} \tag{C.12}$$

No just keep applying equation C.12.

Reduction Rule 6 (RR6) $\langle AB\rangle = \langle A^+B^+ + A^-B^-\rangle$

A^+B^- and A^-B^+ are odd and have no zero grade.

Reduction Rule 7 (RR7) $(\langle B\rangle_2 \cdot a)\cdot b = \langle B\rangle_2 \cdot (a\wedge b)$

Since $\langle B\rangle_2$ is a grade 2 mulitvector it can be written as the linear combination of bivectors $c_i \wedge d_i$. Now apply equation B.10 in Appendix B directly

$$((c_i\wedge d_i)\cdot a)\cdot b = (c_i\wedge d_i)\cdot(a\wedge b)\,. \tag{C.13}$$

In applying the reduction rules one must be carefull of the multivector nature of the multivector derivative. Note that is ψ is a vector then

1. $\nabla\psi$ is an even multivector ($\nabla\psi = \langle\nabla\psi\rangle_0 + \langle\nabla\psi\rangle_2$).
2. ∂_ψ is a vector ($\partial_\psi = \langle\partial_\psi\rangle_1$).
3. $\partial_{\nabla\psi}$ is an even multivector ($\partial_{\nabla\psi} = \langle\partial_{\nabla\psi}\rangle_0 + \langle\partial_{\nabla\psi}\rangle_2$).

If ψ is a spinor (even multivector $\psi = \psi^+$) then

1. $\nabla\psi$ is an odd multivector ($\nabla\psi = (\nabla\psi)^-$).
2. ∂_ψ is an even multivector ($\partial_\psi = (\partial_\psi)^+$).
3. $\partial_{\nabla\psi}$ is an odd multivector ($\partial_{\nabla\psi} = (\partial_{\nabla\psi})^-$).

Appendix D

Curvilinear Coordinates via Matrices

To use matrix algebra to transform from linear to curvilinear coordinates start by defining the matrices

$$\underline{g} = [u_i \cdot u_j] = [g_{ij}] \tag{D.1}$$

$$\overline{g} = \left[u^i \cdot u^j\right] = \left[g^{ij}\right] \tag{D.2}$$

where the u_i's are a set of fixed basis vectors for the vector space. Then define the coordinate vector x by the coordinate functions $x^i(\boldsymbol{\theta})$

$$x(\boldsymbol{\theta}) = x^i(\boldsymbol{\theta})\, u_i \tag{D.3}$$

where $\boldsymbol{\theta}$ is the curvilinear coordinate tuple $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$. In the case of 3-D spherical coordinates $\boldsymbol{\theta} = (r, \theta, \phi)$ where θ is the azimuthal angel measured in the xy plane and ϕ is the elevation angle measured from the z axis. Then the position vector, x, in terms of the coordinate tuple, (r, θ, ϕ), is

$$x = r\left(\sin(\theta)\left(\cos(\phi)\, u_x + \sin(\phi)\, u_y\right) + \cos(\theta)\, u_z\right) \tag{D.4}$$

The unnormalized basis vectors are then given by

$$e_i = \frac{\partial x}{\partial \theta^i}. \tag{D.5}$$

If we represent x as the row matrix

$$\underline{x} = \left[x^i\left(\theta^1, \ldots, \theta^n\right)\right] \tag{D.6}$$

then

$$e_j = \frac{\partial x^i}{\partial \theta^j} u_i = J_{ij} u_i \tag{D.7}$$

and the matrix $\underline{J} = [J_{ij}]$ is

$$\underline{J} = \text{Jacobian}\left(\underline{x}\right). \tag{D.8}$$

Again for spherical coordinates ($\underline{J}^T$ is the transpose of the Jacobian matrix)

$$\underline{J}^T = \begin{bmatrix} \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & \cos(\theta) \\ r\cos(\theta)\cos(\phi) & r\cos(\theta)\sin(\phi) & -r\sin(\theta) \\ -r\sin(\phi) & r\cos(\phi) & 0 \end{bmatrix} \tag{D.9}$$

and

$$e_i = J^T_{ij} u_j. \tag{D.10}$$

The square of the magnitudes of the basis vectors are

$$e_i \cdot e_i = \left(J^T_{im} u_m\right) \cdot \left(J^T_{in} u_n\right) \tag{D.11}$$

$$= J^T_{im} J^T_{in} g_{mn} \tag{D.12}$$

$$\left[e_i^2 \delta_{ij}\right] = \text{diag}\left(\underline{J}^T \underline{g} \underline{J}\right) \tag{D.13}$$

$$\left[|e_i|\, \delta_{ij}\right] = \sqrt{\text{diag}\left(\underline{J}^T \underline{g} \underline{J}\right)} \tag{D.14}$$

. Define the normalized matrix $\underline{\alpha} = [\alpha_{ij}]$ by

$$\alpha_{ij} \equiv \frac{J^T_{ij}}{|e_i|} \tag{D.15}$$

where $|e_i| = \text{sgn}\left(e_i^2\right)\sqrt{|e_i^2|}$ so that $|e_i|$ could be positive or negative depending on the signature of $\underline{g}$ and the actual position vector. We now can write

$$\hat{e}_i = \alpha_{ij} u_j. \tag{D.16}$$

For the reciprocal curvilinear basis vectors $\hat{e}^i$'s we would have

$$\hat{e}^i\left(\boldsymbol{\theta}\right) = \alpha^{ij}\left(\boldsymbol{\theta}\right) u^j. \tag{D.17}$$

To calculate α^{ij} note

$$\hat{e}^i \cdot \hat{e}_j = \alpha^{im} \alpha_{jn} u^m \cdot u_n = \delta_i^j \tag{D.18}$$

$$= \alpha^{im} \alpha_{jn} \delta_n^m = \delta_i^j \tag{D.19}$$

$$= \alpha^{im} \alpha_{jm} = \delta_i^j \tag{D.20}$$

or in matrix terms

$$\overline{\alpha}\underline{\alpha}^T = \mathbf{1} \tag{D.21}$$

$$\overline{\alpha} = \left(\underline{\alpha}^T\right)^{-1} \tag{D.22}$$

$$\overline{\alpha} = \frac{1}{\det\left(\underline{\alpha}^T\right)}\text{adjugate}\left(\underline{\alpha}^T\right). \tag{D.23}$$

We also have

$$G_{ij} = \hat{e}_i \cdot \hat{e}_j \tag{D.24}$$

$$= \alpha_{im}\alpha_{jn}u_m \cdot u_n \tag{D.25}$$

$$= \alpha_{im}g_{mn}\alpha_{jn} \tag{D.26}$$

$$\underline{G} = \underline{\alpha}\underline{g}\underline{\alpha}^T \tag{D.27}$$

$$G^{ij} = \hat{e}^i \cdot \hat{e}^j \tag{D.28}$$

$$= \alpha^{im}\alpha^{jn}u^m \cdot u^n \tag{D.29}$$

$$= \alpha^{im}g^{mn}\alpha^{jn} \tag{D.30}$$

$$\overline{G} = \overline{\alpha}\;\overline{g}\;\overline{\alpha}^T \tag{D.31}$$

The final items required for the curvilinear coordinate transformation are the $\frac{\partial \hat{e}_i}{\partial \theta_k}$'s which are needed for the calculation of geometric derivatives in curvilinear coordinates. The $\frac{\partial \hat{e}_i}{\partial \theta_k}$'s are simply calculated by noting

$$\frac{\partial \hat{e}_i}{\partial \theta_k} = \Gamma_{kij}\hat{e}_j \tag{D.32}$$

$$\hat{e}^m \cdot \frac{\partial \hat{e}_i}{\partial \theta_k} = \Gamma_{kij}\hat{e}^m \cdot \hat{e}_j = \Gamma_{kij}\delta_j^m = \Gamma_{kim} \tag{D.33}$$

$$\Gamma_{kim} = \alpha^{mp}\frac{\partial \alpha_{iq}}{\partial \theta_k}u^p \cdot u_q = \alpha^{mp}\frac{\partial \alpha_{iq}}{\partial \theta_k}\delta_q^p \tag{D.34}$$

$$= \alpha^{mp}\frac{\partial \alpha_{ip}}{\partial \theta_k} \tag{D.35}$$

$$\Gamma_k = \left[\frac{\partial \alpha_{ip}}{\partial \theta_k}\alpha^{mp}\right] = \frac{\partial \underline{\alpha}}{\partial \theta_k}\overline{\alpha}^T. \tag{D.36}$$

Again for spherical coordinates we have for the Γ's

$$\Gamma_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{D.37}$$

$$\Gamma_\theta = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{D.38}$$

$$\Gamma_\phi = \begin{bmatrix} 0 & 0 & \cos(\theta) \\ 0 & 0 & -\sin(\theta) \\ -\cos(\theta) & \sin(\theta) & 0 \end{bmatrix} \tag{D.39}$$

and

$$\begin{array}{ccc} \dfrac{\partial \hat{e}_r}{\partial r} = 0 & \dfrac{\partial \hat{e}_\theta}{\partial r} = 0 & \dfrac{\partial \hat{e}_\phi}{\partial r} = 0 \\ \dfrac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta & \dfrac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r & \dfrac{\partial \hat{e}_\phi}{\partial \theta} = 0 \\ \dfrac{\partial \hat{e}_r}{\partial \phi} = \cos(\theta)\,\hat{e}_\phi & \dfrac{\partial \hat{e}_\theta}{\partial \phi} = -\sin(\theta)\,\hat{e}_\phi & \dfrac{\partial \hat{e}_\phi}{\partial \phi} = -\cos(\theta)\,\hat{e}_r + \sin(\theta)\,\hat{e}_\theta. \end{array} \tag{D.40}$$

Appendix E

Practical Geometric Calculus on Manifolds

The purpose of this appendix is to derive a formulation of geometric algebra/calculus that is suitable for use in a computer algebra system. For this purpose we must reduce or delay the divisions required for the calculation of quatities in the algebra and calculus and to also precalculate those quantities required for the acts of mulitplication and differentiation.

Start with a vector manifold defined by $\boldsymbol{x}\left(\boldsymbol{\theta}\right)$ where $\boldsymbol{\theta}$ is a n-tuple, $\boldsymbol{\theta} = (\theta^1,\dots,\theta^n)$, and $\boldsymbol{x}$ is a vector function of $\boldsymbol{\theta}$ in a vector space of dimension $\geq n$. Then a basis for the tangent space to the manifold is defined by

$$\boldsymbol{e}_i \equiv \frac{\partial \boldsymbol{x}}{\partial \theta^i} \tag{E.1}$$

and the metric tensor by

$$g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j. \tag{E.2}$$

$g_{ij}\left(\boldsymbol{\theta}\right)$ is all that is needed to define the geometric algebra on the tangent space of the manifold.

To define the intrinsic and covariant derivatives we need to calculate the reciprocal basis to the tangent space basis. To do so we use the relations

$$\boldsymbol{e}_i = g_{ij}\boldsymbol{e}^j \tag{E.3}$$

$$\boldsymbol{e}^i = g^{ij}\boldsymbol{e}_j \tag{E.4}$$

where g^{ij} is the inverse of g_{ij}. Additionally, we have from equation 5.167

$$E_n^2 = (\boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n)^2 = (-1)^{\frac{n(n-1)}{2}} \det(g) . \tag{E.5}$$

where $\det(g)$ is the determinant of the metric tensor and $E_n = \boldsymbol{e}_1 \wedge \ldots \wedge \boldsymbol{e}_n$. Now define

$$\bar{\boldsymbol{e}}^i \equiv (-1)^{i-1} (\boldsymbol{e}_1 \wedge \ldots \wedge \breve{\boldsymbol{e}}_j \wedge \ldots \wedge \boldsymbol{e}_n) E_n . \tag{E.6}$$

Then

$$\boldsymbol{e}^i = \frac{\bar{\boldsymbol{e}}^i}{E_n^2} = \frac{\bar{\boldsymbol{e}}^i}{(-1)^{n(n-1)/2} \det(g)}, \tag{E.7}$$

So that calculating $\bar{\boldsymbol{e}}^i$ requires no division.

To define the geometric calculus of the manifold the projections of the derivatives of the tangent vectors onto the tangent space of the manifold is required to define the covariant derivative of a multivector field on the tangent space of the manifold.

The projection of the derivative of the tangent vector into the tangent space is the covariant derivative of the tangent vector, $D_i \boldsymbol{e}_j$ so that

$$D_i \boldsymbol{e}_j \equiv \boldsymbol{e}^k \left(\boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{e}_j}{\partial \theta^i} \right) = \boldsymbol{e}_l g^{lk} \left(\boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{e}_j}{\partial \theta^i} \right) \tag{E.8}$$

and

$$D_i \boldsymbol{e}_j = \frac{1}{(-1)^{n(n-1)/2} \det(g)} \bar{D}_i \boldsymbol{e}_j = \frac{1}{(-1)^{n(n-1)/2} \det(g)} \bar{\boldsymbol{e}}^k \left(\boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{e}_j}{\partial \theta^i} \right) \tag{E.9}$$

so that $\bar{D}_i \boldsymbol{e}_j$ can be calculated without any divide operations (other that those involving caclulating the derivatives of the coefficients of the tangent vector derivatives). We have identified a common divisor for all the covariant derivatives of the tangent vectors.

The next step is to compute $D_i \boldsymbol{e}_j$ we must compute $\boldsymbol{e}_k \cdot \dfrac{\partial \boldsymbol{e}_j}{\partial \theta^i}$. First differentiate g_{ij} with respect to θ^k

$$\frac{\partial g_{ij}}{\partial \theta^k} = \frac{\partial \boldsymbol{e}_i}{\partial \theta^k} \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \cdot \frac{\partial \boldsymbol{e}_j}{\partial \theta^k} = \frac{\partial \boldsymbol{e}_k}{\partial \theta^i} \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \cdot \frac{\partial \boldsymbol{e}_j}{\partial \theta^k} . \tag{E.10}$$

Where the final form of equation E.10 results from the vector manifold relation $\dfrac{\partial \boldsymbol{e}_i}{\partial \theta^j} = \dfrac{\partial \boldsymbol{e}_j}{\partial \theta^i}$. Now cyclicly permute the indices to get

$$\frac{\partial g_{ki}}{\partial \theta^j} = \frac{\partial \boldsymbol{e}_j}{\partial \theta^k} \cdot \boldsymbol{e}_i + \boldsymbol{e}_k \cdot \frac{\partial \boldsymbol{e}_i}{\partial \theta^j} \tag{E.11}$$

$$\frac{\partial g_{jk}}{\partial \theta^i} = \frac{\partial \boldsymbol{e}_i}{\partial \theta^j} \cdot \boldsymbol{e}_k + \boldsymbol{e}_j \cdot \frac{\partial \boldsymbol{e}_k}{\partial \theta^i} . \tag{E.12}$$

Now add equations E.10 to E.11 and subtract E.12 and divide by 2 to get

$$\frac{1}{2}\left(\frac{\partial g_{ij}}{\partial\theta^{k}}+\frac{\partial g_{ki}}{\partial\theta^{j}}-\frac{\partial g_{jk}}{\partial\theta^{i}}\right)=\boldsymbol{e}_{i}\cdot\frac{\partial\boldsymbol{e}_{j}}{\partial\theta^{k}}=\Omega_{ijk}. \tag{E.13}$$

Thus

$$\bar{D}_{i}\boldsymbol{e}_{j}=\Omega_{kji}\bar{\boldsymbol{e}}_{k} \tag{E.14}$$

which can then be pre-calculated for all combinations of basis vectors and coordinates and the results placed in a table or dictionary. Note that while $\partial_k = \dfrac{\partial}{\partial\theta^k}$ is a scalar ooperator, D_k and $\bar{D}_k$ are linear operators, but not scalar operators as can be seen explicitly in equation E.14.

Thus the manifold can be defined either by specifying the embedding function, $\boldsymbol{x}(\boldsymbol{\theta})$, or the metric tensor, $g_{ij}(\boldsymbol{\theta})$. For computational purposes if $\boldsymbol{x}(\boldsymbol{\theta})$ is specified $g_{ij}(\boldsymbol{\theta})$ will be computed from it and Ω_{ijk} computed from $g_{ij}(\boldsymbol{\theta})$.

Now consider an r-grade multivector functon F_r on the manifold given by (using the summation convention)

$$F_{r}=F^{i_1\dots i_r}\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r} \tag{E.15}$$

where both $F^{i_1\dots i_r}$ and the $\boldsymbol{e}_{i_j}$ can be functions of $\boldsymbol{\theta}$, the coordinates and that $i_1 < \cdots < i_l < \cdots < i_r \;\forall\; 1 \geq i_l \geq n$. The the intrinsic derivative of F_r is defined by (summation convention again for repeated indices)

$$\partial F_{r}\equiv\boldsymbol{e}^{j}\partial_{j}F_{r} \tag{E.16}$$

$$=\partial_{j}F^{i_1\dots i_r}\boldsymbol{e}^{j}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right)+F^{i_1\dots i_r}\boldsymbol{e}^{j}\sum_{l=1}^{r}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\partial_{j}\boldsymbol{e}_{i_l}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right). \tag{E.17}$$

The covariant dervative, D is defined as the projection of the intrinsic derivative onto the tangent space of the manifold (the tangent space can be defined by the pseudo scalar E_n)

$$DF_{r}\equiv\mathcal{P}_{E_n}\left(\partial F_{r}\right). \tag{E.18}$$

Thus

$$DF_{r}=\partial_{j}F^{i_1\dots i_r}\boldsymbol{e}^{j}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right)+F^{i_1\dots i_r}\boldsymbol{e}^{j}\sum_{l=1}^{r}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge D_{j}\boldsymbol{e}_{i_l}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right). \tag{E.19}$$

For computational purposes

$$\begin{aligned} DF_r =& \frac{1}{(-1)^{n(n-1)/2}\det(g)}\partial_j F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j \left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right) \\ &+ \frac{1}{\det(g)^2}F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j\sum_{l=1}^{r}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\bar{D}_j\boldsymbol{e}_{i_l}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right) \quad\text{(E.20)} \\ =& \frac{1}{(-1)^{n(n-1)/2}\det(g)}\partial_j F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j \left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right) \\ &+ \frac{1}{\det(g)^2}F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j\sum_{l=1}^{r}\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\left(\Omega_{klj}\bar{\boldsymbol{e}}^k\right)\wedge\dots\wedge\boldsymbol{e}_{i_r}\right) \quad\text{(E.21)} \end{aligned}$$

Again for computational purposes $\bar{D}_j\boldsymbol{e}_k$ will be calculated in terms of a linear combination of $\boldsymbol{e}_j$'s and then the connection multivectors computed from ordered combinations of $i_1\dots i_r$ $1\le r\le n$

$$C\{i_1\dots i_r\} \equiv \bar{\boldsymbol{e}}^j\sum_{l=1}^{r}\boldsymbol{e}_{i_1}\wedge\dots\wedge\ \left(\bar{D}_j\boldsymbol{e}_{i_l}\right)\wedge\dots\wedge\boldsymbol{e}_{i_r}, \quad\text{(E.22)}$$

$$\begin{aligned} DF_r =& \frac{1}{(-1)^{n(n-1)/2}\det(g)}\partial_j F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right) \\ &+ \frac{1}{\det(g)^2}F^{i_1\dots i_r}C\{i_1\dots i_r\}. \quad\text{(E.23)} \end{aligned}$$

It is probably best when calculating DF_r (or $DF = D\langle F\rangle_0 + \dots + D\langle F\rangle_n$) to have the option of returning $(-1)^{n(n-1)/2}\,\partial_j F^{i_1\dots i_r}\bar{\boldsymbol{e}}^j\left(\boldsymbol{e}_{i_1}\wedge\dots\wedge\boldsymbol{e}_{i_r}\right)$ and $F^{i_1\dots i_r}C\{i_1\dots i_r\}$ separately since the $\det(g)$ scaling for each term is different.

To calculate the outer and inner covariant dervatives use the formulas

$$D\wedge F = \langle D\langle F\rangle_0\rangle_1 + \dots + \left\langle D\langle F\rangle_{n-1}\right\rangle_n, \quad\text{(E.24)}$$

$$D\cdot F = \langle D\langle F\rangle_1\rangle_0 + \dots + \langle D\langle F\rangle_n\rangle_{n-1}. \quad\text{(E.25)}$$

Appendix F

Direct Sum of Vector Spaces

Let U and V be vector spaces with $\dim(U) = n$ and $\dim(V) = m$. Additionally, let the sets of vectors $\{\boldsymbol{u}_i\}$ and $\{\boldsymbol{v}_j\}$ be basis sets for U and V respectively. The direct sum, $U \oplus V$, is defined to be

$$U \oplus V \equiv \{(\boldsymbol{u}, \boldsymbol{v}) \,|\, \boldsymbol{u} \in U, \boldsymbol{v} \in V\}, \tag{F.1}$$

where vector addition and scalar multiplication are defined by for all $(\boldsymbol{a}_1, \boldsymbol{b}_1), (\boldsymbol{a}_2, \boldsymbol{b}_2) \in U \oplus V$ and $\alpha \in \Re$

$$(\boldsymbol{a}_1, \boldsymbol{b}_1) + (\boldsymbol{a}_2, \boldsymbol{b}_2) \equiv (\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{b}_1 + \boldsymbol{b}_2) \tag{F.2}$$

$$\alpha(\boldsymbol{a}_1, \boldsymbol{b}_1) \equiv (\alpha\boldsymbol{a}_1, \alpha\boldsymbol{b}_1). \tag{F.3}$$

Now define the maps $i_U : U \rightarrow U \oplus V$ and $i_V : V \rightarrow U \oplus V$ by

$$i_U(\boldsymbol{u}) \equiv (\boldsymbol{u}, \boldsymbol{0}) \tag{F.4}$$

$$i_V(\boldsymbol{v}) \equiv (\boldsymbol{0}, \boldsymbol{v}). \tag{F.5}$$

Thus $(\boldsymbol{u}, \boldsymbol{v}) = i_U(\boldsymbol{u}) + i_V(\boldsymbol{v})$ and the set $\{i_U(\boldsymbol{u}_i), i_V(\boldsymbol{v}_j) \,|\, 0 < i \le n, 0 < j \le m\}$ form a basis for $U \oplus V$ so that if $\boldsymbol{x} \in U \oplus V$ then $(\alpha^i, \beta^j \in \Re)$ and using the Einstein summation convention

$$\boldsymbol{x} = \alpha^i i_U(\boldsymbol{u}_i) + \beta^j i_V(\boldsymbol{v}_j). \tag{F.6}$$

If from context we known that $\boldsymbol{x} \in U \oplus V$ and we are expanding $\boldsymbol{x}$ in terms of a basis of $U \oplus V$ we will write as a notational convenience $\boldsymbol{u}_i$ for $i_U(\boldsymbol{u}_i)$ and $\boldsymbol{v}_i$ for $i_V(\boldsymbol{v}_i)$ so that we may write

$$\boldsymbol{x} = \alpha^i \boldsymbol{u}_i + \beta^j \boldsymbol{v}_j. \tag{F.7}$$

Also for notational convenience denote for any $\boldsymbol{x} \in U \oplus V$, $x_U = i_U(\boldsymbol{x})$ and $x_V = i_V(\boldsymbol{x})$.

Likewise we define the mappings $p_U : U \oplus V \to U$ and $p_V : U \oplus V \to V$ by

$$p_U((\boldsymbol{u}, \boldsymbol{v})) \equiv \boldsymbol{u} \tag{F.8}$$

$$p_V((\boldsymbol{u}, \boldsymbol{v})) \equiv \boldsymbol{v}. \tag{F.9}$$

For notational convenience if $p_V((\boldsymbol{u}, \boldsymbol{v})) = \boldsymbol{0}$ we write $\boldsymbol{u} = p_U((\boldsymbol{u}, \boldsymbol{v}))$ and if $p_U((\boldsymbol{u}, \boldsymbol{v})) = \boldsymbol{0}$ we write $\boldsymbol{v} = p_V((\boldsymbol{u}, \boldsymbol{v}))$. We always make the identifications

$$\begin{aligned} \boldsymbol{u} &\leftrightarrow (\boldsymbol{u}, \boldsymbol{0}) \\ \boldsymbol{v} &\leftrightarrow (\boldsymbol{0}, \boldsymbol{v}). \end{aligned}$$

Which one to use will be clear from context or will be explicitly identified.

One final notational convenience is that for $(\boldsymbol{u}, \boldsymbol{v}) \in U \oplus V$ we make the equivalence

$$(\boldsymbol{u}, \boldsymbol{v}) \leftrightarrow \boldsymbol{u} + \boldsymbol{v}.$$

Appendix G

sympy/galgebra evaluation of $GL(n, \Re)$ structure constants

The structure constants of $GL(n, \Re)$ can be calculated using the *sympy* python computer algebra system wiht the latest *galgebra* modules (`https://github.com/brombo/sympy`). The python program used for the calculation is shown below:

```
#Lie Algebras
from sympy import symbols
from sympy.galgebra.ga import Ga
from sympy.galgebra.mv import Com
from sympy.galgebra.printer import Format, xpdf

#General Linear Group E generators
def E(i,j):
    global e,eb
    B = e[i]*e[j] - eb[i]*eb[j]
    Bstr = 'E_{' + i + j + '} '
    print '%' + Bstr + ' =',B
    return Bstr, B

#General Linear Group F generators
def F(i,j):
    global e,eb
    B = e[i]*eb[j] - eb[i]*e[j]
    Bstr = 'F_{' + i + j + '} '
    print '%' + Bstr + ' =',B
    return Bstr, B

#General Linear Group K generators
def K(i):
    global e,eb
    B = e[i]*eb[i]
    Bstr = 'K_{' + i + '} '
    print '%' + Bstr + ' =',B
    return Bstr, B

#Print Commutator
def ComP(A,B):
    AxB = Com(A[1],B[1])
    AxBstr = '%' + A[0] + r' \times ' + B[0] + ' ='
    print AxBstr, AxB
    return

Format()

(glg,ei,ej,em,en,ebi,ebj,ebm,ebn) = Ga.build(r'e_i e_j e_k e_l \bar{e}_i \bar

e = {'i':ei,'j':ej,'k':em,'l':en}
eb = {'i':ebi,'j':ebj,'k':ebm,'l':ebn}

print r'#\centerline{General Linear Group of Order $n$\newline}'
print r'#Lie Algebra Generators: $1\le i < j \le n$ and $1 \le i < l \le n$'

#Calculate Lie Algebra Generators
Eij = E('i','j')
Fij = F('i','j')
Ki = K('i')
Eil = E('i','l')
Fil = F('i','l')

print r'#Non Zero Commutators'
```

```
#Calculate and Print Non-Zero Generator Commutators
ComP(Eij,Fij)
ComP(Eij,Ki)
ComP(Fij,Ki)
ComP(Eij,Eil)
ComP(Fij,Fil)
ComP(Fij,Eil)

xpdf(paper='letter',pt='12pt',debug=True,prog=True)
```

Only those commutators that share one or two indices are calculated (all others are zero). The LaTeX output of the program follows:

General Linear Group of Order n

Lie Algebra Generators: $1 \leq i < j \leq n$ and $1 \leq i < l \leq n$

$$E_{ij} = e_i \wedge e_j - \bar{e}_i \wedge \bar{e}_j$$

$$F_{ij} = e_i \wedge \bar{e}_j + e_j \wedge \bar{e}_i$$

$$K_i = e_i \wedge \bar{e}_i$$

$$E_{il} = e_i \wedge e_l - \bar{e}_i \wedge \bar{e}_l$$

$$F_{il} = e_i \wedge \bar{e}_l + e_l \wedge \bar{e}_i$$

Non Zero Commutators

$$E_{ij} \times F_{ij} = 2e_i \wedge \bar{e}_i - 2e_j \wedge \bar{e}_j$$

$$E_{ij} \times K_i = -e_i \wedge \bar{e}_j - e_j \wedge \bar{e}_i$$

$$F_{ij} \times K_i = -e_i \wedge e_j + \bar{e}_i \wedge \bar{e}_j$$

$$E_{ij} \times E_{il} = -e_j \wedge e_l + \bar{e}_j \wedge \bar{e}_l$$

$$F_{ij} \times F_{il} = e_j \wedge e_l - \bar{e}_j \wedge \bar{e}_l$$

$$F_{ij} \times E_{il} = e_j \wedge \bar{e}_l + e_l \wedge \bar{e}_j$$

The program does not completely determine the structure constants since the *galgebra* module cannot currently solve a bivector equation. One must inspect the calculated commutator to see what the linear expansion of the commuatator is in terms of the Lie algebra generators. For this case the answer is:

$$E_{ij} \times F_{ij} = 2\left(K_i - K_j\right) \tag{G.1}$$

$$E_{ij} \times K_i = -F_{ij} \tag{G.2}$$

$$F_{ij} \times K_i = -E_{ij} \tag{G.3}$$

$$E_{ij} \times E_{il} = -E_{jl} \tag{G.4}$$

$$F_{ij} \times F_{il} = E_{jl} \tag{G.5}$$

$$F_{ij} \times E_{il} = F_{jl}. \tag{G.6}$$

Appendix H

Blade Orientation Theorem

A blade only depends on the relative orientation of the vectors in the plane defined by the blade. Since any blade can be defined by the geometric product of two orthogonal vectors let them be $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$. Then any two vectors in the plane can be define by:

$$a = a_x\boldsymbol{e}_x + a_y\boldsymbol{e}_y \tag{H.1}$$

$$b = b_x\boldsymbol{e}_x + b_y\boldsymbol{e}_y \tag{H.2}$$

and any rotor in the plane by

$$R = ab = (a \cdot b) + (a_xb_y - a_yb_x)\,\boldsymbol{e}_x\boldsymbol{e}_y \tag{H.3}$$

as long as

$$RR^\dagger = 1 \tag{H.4}$$

but

$$R\boldsymbol{e}_x\boldsymbol{e}_y = \boldsymbol{e}_x\boldsymbol{e}_yR \tag{H.5}$$

and

$$R\boldsymbol{e}_xR^\dagger R\boldsymbol{e}_yR^\dagger = R\boldsymbol{e}_x\boldsymbol{e}_yR^\dagger = \boldsymbol{e}_x\boldsymbol{e}_yRR^\dagger = \boldsymbol{e}_x\boldsymbol{e}_y \tag{H.6}$$

and absolute orientations of $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ does not matter for $\boldsymbol{e}_x\boldsymbol{e}_y$.

Appendix I

Case Study of a Manifold for a Model Universe

We wish to construct a spatially curved closed isotropic Minkowski space with 1, 2, and 3 spatial dimensions.

To do this consider a Minkowski space with one time dimension ($e_0^2 = 1$) and 2, 3, or 4 spatial dimensions (unit vector squares to -1) as indicated below

$$e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = -e_4^2 = 1 \tag{I.1}$$

The vector manifolds will designated by

Vector Function	Spatial Dimensions	Coordinates	Components
$X^{(1)}$	1	τ, ρ	e_0, e_1, e_2
$X^{(2)}$	2	τ, ρ, θ	e_0, e_1, e_2, e_3
$X^{(3)}$	3	τ, ρ, θ, ϕ	e_0, e_1, e_2, e_3, e_4

The condition that τ does parametrize the time coordinate of the manifold is that

$$e_\tau \cdot e_\tau = \frac{\partial X^{(i)}}{\partial \tau} \cdot \frac{\partial X^{(i)}}{\partial \tau} = 1 \tag{I.2}$$

since then the time coordinate is given by (all spatial coordinates are fixed)

$$\int \sqrt{e_\tau \cdot e_\tau}\, d\tau = \tau \tag{I.3}$$

Then the manifolds in 1, 2, and 3 spatial dimensions are defined by

$$X^{(1)} = t(\tau)\, e_0 + r(\tau)\left(\cos\left(\frac{\rho}{r}\right)e_1 + \sin\left(\frac{\rho}{r}\right)e_2\right) \tag{I.4}$$

$$X^{(2)} = t(\tau)\, e_0 + r(\tau)\left(\cos\left(\frac{\rho}{r}\right)e_1 + \sin\left(\frac{\rho}{r}\right)(\cos\theta e_2 + \sin\theta e_3)\right) \tag{I.5}$$

$$\begin{aligned} X^{(3)} = {} & t(\tau)\, e_0 + \\ & r(\tau)\left(\cos\left(\frac{\rho}{r}\right)e_1 + \sin\left(\frac{\rho}{r}\right)(\cos\theta e_2 + \sin\theta\,(\cos\phi e_3 + \sin\phi e_4))\right) \end{aligned} \tag{I.6}$$

Where $r(\tau)$ is the spatial radius of the manifolds. In order for equation I.2 to be satisfied for all the three manifolds we must have

$$e_\tau \cdot e_\tau = \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 = 1 \tag{I.7}$$

For the specific case of $r(\tau) = \frac{1}{2}\tau^2$ the derivatives and $t(\tau)$ are given by

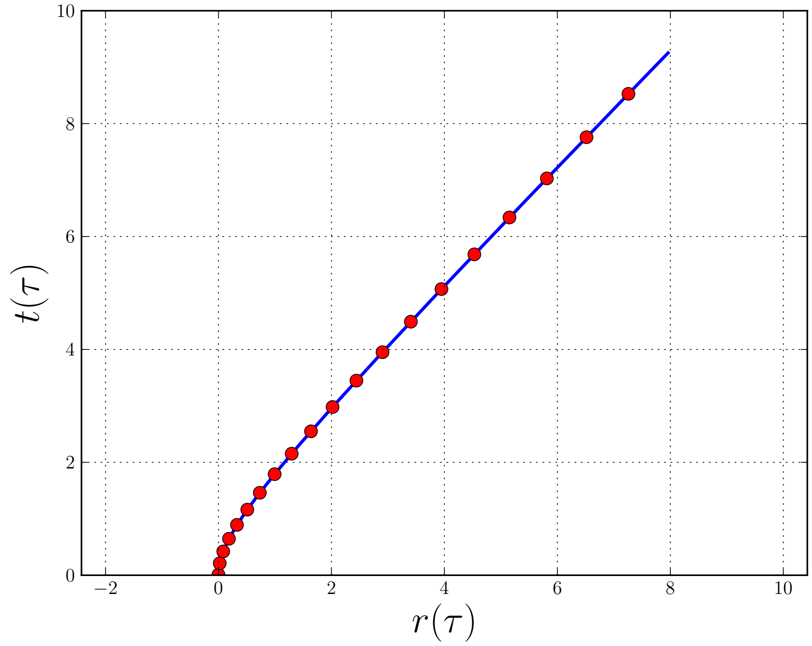

Figure I.1: Expansion Radius of Model Universe, $r(\tau)$, where τ is the time coordinate.

$$\begin{aligned}\frac{dr}{d\tau} &= \tau \\ \frac{dt}{d\tau} &= \sqrt{1+\tau^2} \\ t\left(\tau\right) &= \frac{1}{2}\left(\tau\sqrt{1+\tau^2}+\sinh^{-1}\tau\right)\end{aligned}$$

Spatial coordinates for the model universes are given by ρ a spatial radius and the angular coordinates θ and ϕ if required. A visualization of the 1-D spatial dimensional manifold is shown in figure I.2. e_τ and e_ρ are

$$e_\tau = \frac{\partial X^{(1)}}{\partial \tau} \quad \text{and} \quad e_\rho = \frac{\partial X^{(1)}}{\partial \rho} \tag{I.8}$$

and metric tensors for the three cases are

$$g^{(i)}_{\mu\nu} = \frac{\partial X^{(i)}}{\partial \mu}\cdot\frac{\partial X^{(i)}}{\partial \nu} \tag{I.9}$$

where $\mu, \nu = \{\tau, \rho, \theta, \phi\}$. If we define $h\left(\tau,\rho\right)$ as

$$h\left(\tau,\rho\right) = \frac{dr}{d\tau}\frac{\rho}{r\left(\tau\right)} \tag{I.10}$$

Then the differential arclength given by the metric tensor, $g_{\mu\nu}$, is

$$\left(ds\right)^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{I.11}$$

and the metric tensors for the three cases are (using sympy to do the algebra)

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h \\ h & -1 \end{pmatrix} \tag{I.12}$$

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 \\ h & -1 & 0 \\ 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\right)^2 \end{pmatrix} \tag{I.13}$$

$$g^{(3)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 & 0 \\ h & -1 & 0 & 0 \\ 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\right)^2 & 0 \\ 0 & 0 & 0 & -\left(r\sin\left(\frac{\rho}{r}\right)\sin\theta\right)^2 \end{pmatrix}. \tag{I.14}$$

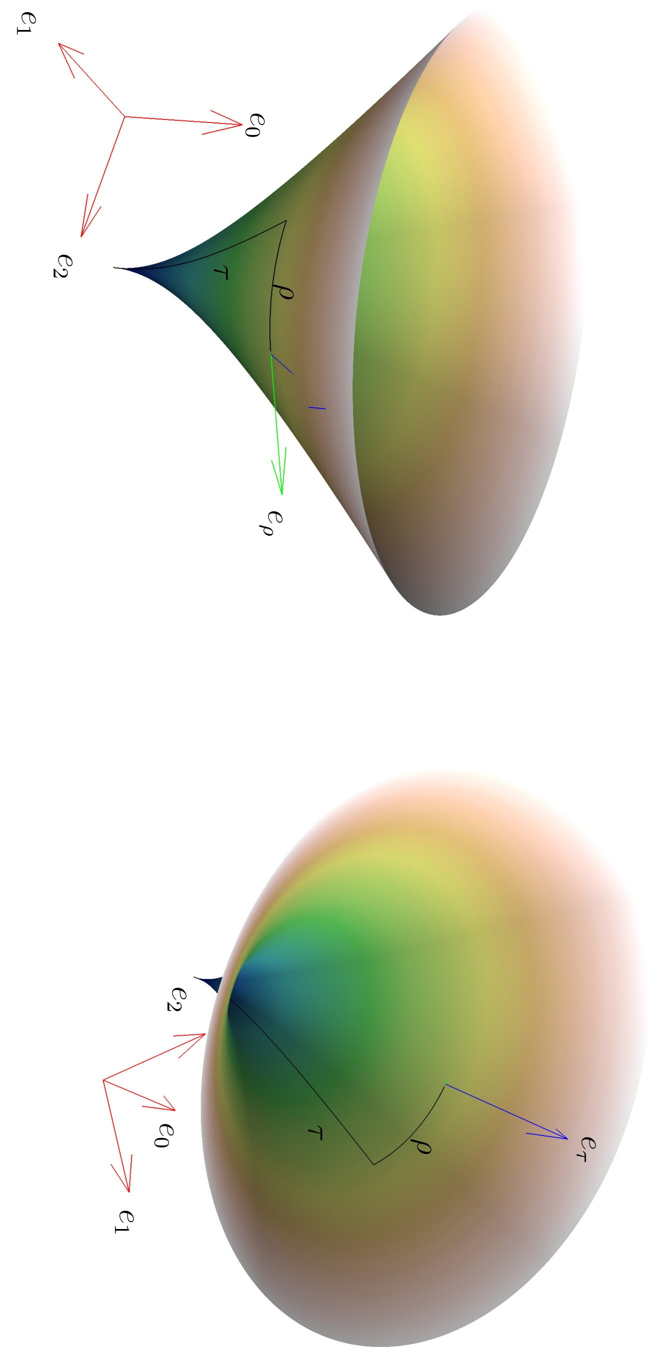

Figure I.2: 1D Universe

If we renormalize e_θ and e_ϕ to be unit vectors

$$e'_\theta = \frac{e_\theta}{\left| r \sin\left(\frac{\rho}{r}\right) \right|} \tag{I.15}$$

$$e'_\phi = \frac{e_\phi}{\left| r \sin\left(\frac{\rho}{r}\right) \sin\theta \right|} \tag{I.16}$$

The metric tensors $g_{\mu\nu}^{(2)}$ and $g_{\mu\nu}^{(3)}$ become

$$g'^{(2)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 \\ h & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{I.17}$$

$$g'^{(3)}_{\mu\nu} = \begin{pmatrix} 1-h^2 & h & 0 & 0 \\ h & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{I.18}$$

and

$$\det\left(g^{(1)}_{\mu\nu}\right) = \det\left(g'^{(2)}_{\mu\nu}\right) = \det\left(g'^{(3)}_{\mu\nu}\right) = -1 = I^2 \tag{I.19}$$

For the 1-Dimensional space the differential arc length is

$$(ds)^2 = \left(1-h^2\right)(d\tau)^2 + 2h d\tau d\rho + (d\rho)^2 \tag{I.20}$$

so that for the light cone, $ds = 0$, we have the differential equation

$$1 - h^2 + 2h\frac{d\rho}{d\tau} + \left(\frac{d\rho}{d\tau}\right)^2 = 0 \tag{I.21}$$

Note that this equation also applies to the 2 and 3 dimensional case if we set $\frac{d\theta}{d\tau} = \frac{d\phi}{d\tau} = 0$. Solving for $\frac{d\rho}{d\tau}$ gives

$$\frac{d\rho}{d\tau} = h \pm 1 = \frac{1}{r}\frac{dr}{d\tau}\rho \pm 1 = \left(\frac{d}{d\tau}\ln(r)\right)\rho \pm 1 \tag{I.22}$$

Using the integration factor for linear first order differential equations the solution to equation I.22 is $(\rho(\tau_0) = 0)$

$$\rho(\tau) = \pm r(\tau) \int_{\tau_0}^{\tau} \frac{d\tau'}{r(\tau')} \tag{I.23}$$

Note that $r\left(\tau\right)$ and $\alpha r\left(\tau\right)$ have the same solution $\rho\left(\tau\right)$. Now consider the case that $r\left(\tau\right)=\tau^{\eta}$, then

$$\rho\left(\tau\right)=\pm\tau^{\eta}\int_{\tau_0}^{\tau}\left(\tau'\right)^{-\eta}d\tau'=\left\{\begin{array}{ll}\eta=1, & \pm\tau\ln\left(\dfrac{\tau}{\tau_0}\right)\\ \eta\neq 1, & \dfrac{\pm 1}{1-\eta}\left(\tau-\tau_0\left(\dfrac{\tau}{\tau_0}\right)^{\eta}\right)\end{array}\right\} \tag{I.24}$$

Typical $\rho\left(\tau\right)$'s for various $-1.5\leq\eta\leq 1.5$ are shown in the light cone plot. If $\eta>0$ the speed of light is greater than c in flat space. If $\eta<0$ the speed of light is less than c in flat space. Note that if the universe is curved, but not expanding or contracting the speed of light is the same as c in flat space.

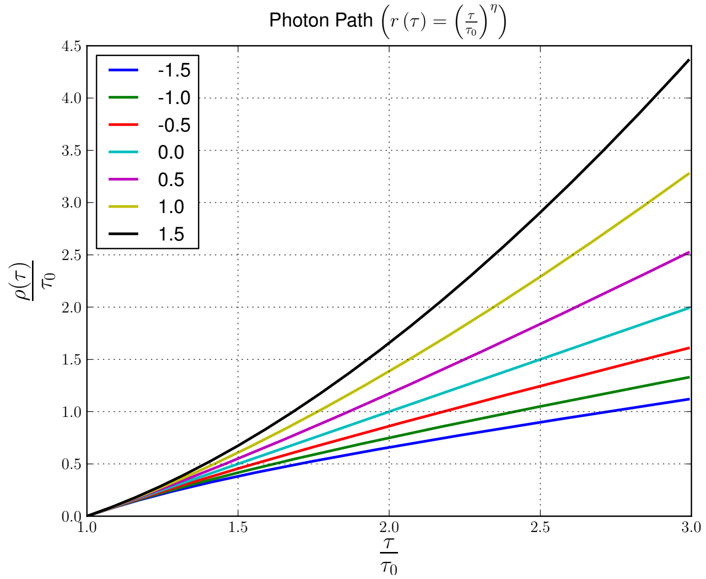

Figure I.3: Lightcone in Curved Space

For a time periodic universe $r\left(\tau\right)=\sin\left(\dfrac{\tau}{T}\right)$ where T is twice the period of the universe and $\rho\left(\tau_0\right)=0$, then

$$\rho\left(\tau\right)=T\sin\left(\frac{\tau}{T}\right)\ln\left|\frac{\csc\left(\dfrac{\tau_0}{T}\right)+\cot\left(\dfrac{\tau_0}{T}\right)}{\csc\left(\dfrac{\tau}{T}\right)+\cot\left(\dfrac{\tau}{T}\right)}\right| \tag{I.25}$$

The plot of equation I.25 is shown in figure I.4

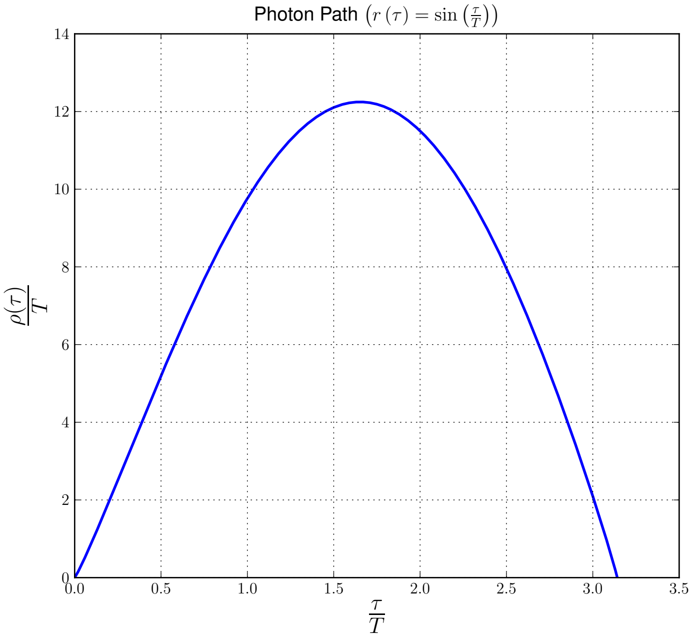

Figure I.4: Periodic Light Cone

I.1 The Edge of Known Space

Another kinematic question to answer is under what conditions light cannot access parts of the universe. The critical quantity is

$$\lambda\left(\tau\right)=\frac{\rho\left(\tau\right)}{\vartheta r\left(\tau\right)} \tag{I.26}$$

where ϑ (the angular distance around the universe restricted to $0\le\vartheta\le\pi$) is the measure of how far from the observer you are. Since the universe is spatially periodic the maximum value of ϑ is π. If $\lambda\left(\tau\right)\ge 1$ for some finite τ you can access the distance defined by ϑ. Substituting equation I.23 into equation I.26 gives

$$\lambda\left(\tau\right)=\frac{1}{\vartheta}\int_{\tau_0}^{\tau}\frac{d\tau'}{r\left(\tau'\right)} \tag{I.27}$$

so that the connection condition is

$$\int_{\tau_0}^{\tau}\frac{d\tau'}{r\left(\tau'\right)}\ge\vartheta. \tag{I.28}$$

First consider a linear expansion model of the form

$$r\left(\tau\right)=r_0\left(1+\alpha\left(\frac{\tau}{\tau_0}-1\right)\right) \tag{I.29}$$

where $r\left(\tau_0\right)=r_0$. Then

$$\frac{\tau}{\tau_0}\ge\frac{1}{\alpha}e^{\alpha\vartheta\left(\frac{r_0}{\tau_0}\right)}-1 \tag{I.30}$$

In a linearly expanding universe the photon time of flight increases exponentially with distance. Now consider super-linear expansion of the form ($\eta>1$)

$$r\left(\tau\right)=r_0\left(1+\alpha\left(\frac{\tau}{\tau_0}-1\right)^{\eta}\right) \tag{I.31}$$

Then

$$\frac{\tau_0}{r_0}\alpha^{-\frac{1}{\eta}}\int_0^{\alpha^{\frac{1}{\eta}}\left(\frac{\tau}{\tau_0}-1\right)}\frac{d\mu}{1+\mu^{\eta}}\ge\vartheta, \tag{I.32}$$

but the integral in equation I.32 does not have a closed form solution unless we let $\tau\to\infty$. In that case[1] we can write

$$\frac{1}{\eta}\alpha^{-\frac{1}{\eta}}\csc\left(\frac{\pi}{\eta}\right)\ge\left(\frac{\vartheta}{\pi}\right)\left(\frac{r_0}{\tau_0}\right) \tag{I.33}$$

A contour plot of the left side of equation I.33 is shown below

---

[1] $\int_0^{\infty}\frac{dx}{1+x^{\eta}}$ is 3.241-2 in "Gradshteyn and Ryzhik"

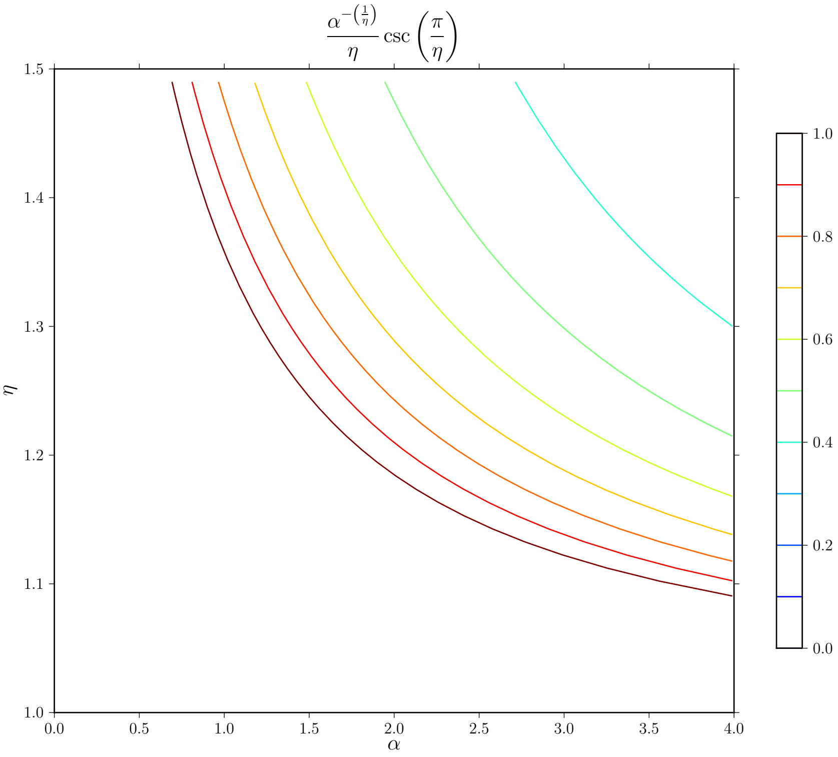

Figure I.5: Photon Exclusion Zone

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The right side of equation I.33, $\left(\frac{\vartheta}{\pi}\right)\left(\frac{r_0}{\tau_0}\right)$, is interpreted as follows -

1. $\frac{\vartheta}{\pi}$ is the fractional distance around the closed spatially periodic universe. $\frac{\vartheta}{\pi} = 1$ is as far as one can go before the distance from the observer starts to decrease.
2. $\frac{r_0}{\tau_0}$ is a measure of inflation. Immediately after an inflationary epoch $\frac{r_0}{\tau_0} >> 1$.

Thus equation I.33 determined the maximum distance $\frac{\vartheta}{\pi}$ that a photon can propagate in a finite amount of time.

Another question to consider is under what conditions $\frac{r_0}{\tau_0}$ will increase as τ_0 increases or when will the following be true

$$\frac{r(\tau)}{\tau} \geq \frac{r(\tau_0)}{\tau_0}. \tag{I.34}$$

Equation I.34 is equivalent to

$$\alpha\left(\frac{\tau}{\tau_0} - 1\right)^{\eta-1} \geq 1 \tag{I.35}$$

so that if $\eta > 1$ then $\frac{r(\tau_0)}{\tau_0}$ will eventually grow as τ_0 increases.

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